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A concise geometric proof of the three distance theorem

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ABSTRACT. The *three distance theorem* states that for any given irrational number α and a natural number n, when the interval $(0, 1)$ is divided into $n + 1$ subintervals by integer multiples of α, namely, $\{0\}$, $\{\alpha\}$, $\{2\alpha\}$, ..., $\{n\alpha\}$, then each subinterval is limited to at most three different lengths. Steinhaus conjectured this theorem in the 1950s, and many researchers have given various proofs since then.
This paper aims to improve the perspective by showing a two-dimensional map which tells how the unit interval is divided by continuously changing α, and provide a concise proof of the theorem. By illustrating this proof through geometric visualizations, we offer a clearer and more intuitive understanding of the underlying principles and relationships. The approach not only reinforces the classical results but also paves the way for new insights and applications in the study of irrational numbers and their properties.

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1. Introduction

The historical accounts suggest that Pythagorean school discovered that the sounds produced by strings resonate harmoniously when the ratio of string lengths is $3 : 2$ (for example, [**?**], [**?**], [**?**]). Building on this, we can observe the fact that the three distance theorem can be naturally derived by further exploring the findings from the proportional relationships behind harmonic intervals as follows.

Starting with the note F and successively multiplying the string length by $2/3$ (equivalent to multiplying frequency by $3/2$), and doubling it when less than half the original length, we generate the Pythagorean temperament: $F \to C \to G \to D \to A \to E \to B$. Figure **??** illustrates how this temperament subdivides an octave in

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frequency. By multiplying $\log_2 3$ by rank numbers $1, 2, 3, \ldots$, and marking the fractional parts (since 1 and 2 correspond to 0 and 1 on a logarithmic scale with base 2), we observe the formation of the Pythagorean temperament.

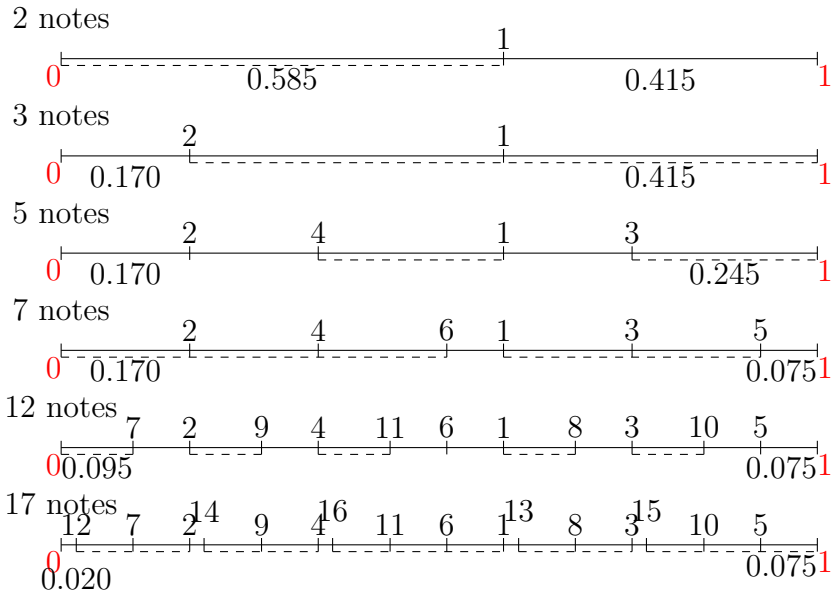


Figure 1: Formation of Pythagorean temperament.

Arranging these values in ascending order and examining the differences between adjacent values, we find that the differences are limited to two or three values. Figure **??** shows the divisions for two different values of difference, in order of increasing number of divisions. Temperaments with 5, 7, or 12 divisions of an octave are typical in traditional and contemporary music.

We can also see that the number of divisions with two difference values corresponds to the denominators of convergents obtained by expanding $\log_2 3$ into continued fractions and that the interval adjacent to 0 differs from that adjacent to 1. Furthermore, the difference in rank numbers of adjacent values equals one of the two types of interval numbers modulo n, with the sign differing based on whether the difference is a large or small interval. This relationship holds at both ends of an octave if the rank number corresponding to 0 and 1 is set to 0.

These observations lead naturally to the three distance theorem since it is understood that the above holds true even if $\log_2 3$ is replaced by an arbitrary, irrational number, given a continued fraction expansion. This fact can also be interpreted as an expression of regularity of the image of a finite arithmetic progression with the initial term 0 in the projection from the set of real numbers $\mathbb{R}$ to the unit interval $\mathbb{R}/\mathbb{Z}(= [0, 1))$. Steinhaus formulated this as a conjecture only in the second half of the 20th century, and many researchers have provided various proofs since then.

Here we give a brief survey of the different approaches adopted by various authors to prove the three distance theorem and its dual, the three gap theorem.

Previous proofs of the three distance theorem have shown that for a given α, the lengths of line segments $\{i\alpha\}$ dividing the unit interval are determined by $\min(\{i\alpha\})$ and $\max(1 - \{i\alpha\})$. Throughout the 20th century, proofs were primarily based on elementary number theory concepts like continued fractions and combinatorics, as seen in works like [?], [?], [?], [?], [?]. Alessandri and Berthé provided a survey paper [?] summarizing these efforts.

In the 21st century, researchers adopted contemporary methodologies. Biringer and Schmidt [?] explored the Riemannian manifold, while Haynes, Koivusalo, Walton, and Sadun [?] examined CW-complexes. Marklof and Strömbergsson [?] used the modular group with lattice structures, and Taha [?] utilized properties of Farey fractions. Bockiting-Conrad, Kashina, Petersen, and Tenner [?] employed Sós permutations, and Nagata and Takei [?] explored a two-dimensional version.

Our contribution lies in providing a new geometric perspective on the three distance theorem. We demonstrate how the unit interval is divided by continuously varying α, presenting a two-dimensional map that visualizes this process. This approach not only simplifies the understanding of the theorem but also provides an intuitive visual representation of the relationships involved. Our method unifies the understanding of the theorem for both irrational and rational numbers, thereby offering a comprehensive insight into the behavior of points on the unit interval.

Traditionally, proofs have discussed α as a fixed specific value. However, our approach visualizes the division of the unit interval for varying α, showing how it changes for different α values. This progression enables a shift from understanding discrete sets of values (specific rational and irrational α) to a comprehensive understanding over the continuous range (all real numbers α between 0 and 1). This method reinforces the classical results and opens new avenues for a comprehensive understanding of the theorem across the entire set of real numbers, including both rational and irrational numbers.

2. Preparation

Let $\lfloor x \rfloor$ ($x \in \mathbb{R}$) denote the largest integer not exceeding x, and $\{x\}$ the fractional part of $x \in \mathbb{R}$, i.e., $\{x\} \coloneqq x - \lfloor x \rfloor$.

The *Farey sequence* F_n for any positive integer n consists of irreducible rational numbers $\frac{a}{b}$ with $0 \leq a \leq b \leq n$ and $\gcd(a, b) = 1$, arranged in increasing order. Consecutive fractions $(\frac{a}{b}, \frac{c}{d})$ in F_n form a *Farey pair*. The *mediant* of two fractions $\frac{a}{b}$

and $\frac{c}{d}$ is defined as $\frac{a+c}{b+d}$. In the Farey sequence F_n for n, interpolating the mediants of the Farey pairs where the sum of the denominators is $n+1$ generates F_{n+1}, the Farey sequence for $n+1$. An interval $(\frac{a}{b}, \frac{c}{d})$ is called a *Farey interval* if $\frac{a}{b}$ and $\frac{c}{d}$ are a Farey pair in some F_n.

Next, we define the *unit interval integral partition* of order n (U_n) as a two-dimensional region with $\alpha \in (0,1)$ on the horizontal axis and height $h \in [0,1)$ on the vertical axis, with lines $h = \{i\,\alpha\},\ (0 < \alpha < 1,\ i = 1, \ldots, n)$. Once α is determined, the line segment representing $h \in [0,1)$ is divided into, at most $n+1$ subintervals. For convenience, separated individual intervals are assumed to be semi-open intervals, closed to the left and open to the right. Figure **??** shows the case $n = 3$.

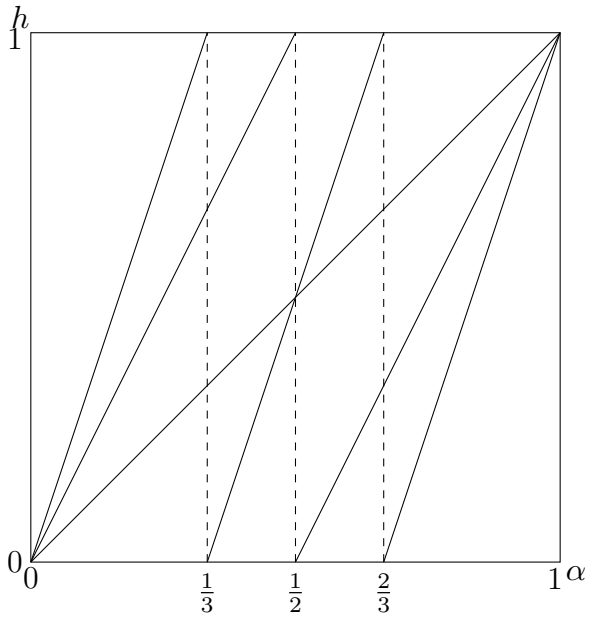


Figure 2: Unit interval integral partition of order 3 (U_3): $h = \{i\alpha\},\ (0 < \alpha < 1,\ i = 1, 2, 3)$.

Similar ideas to our unit interval integral partition were introduced by Berstel and Pocchiola [**?**] to investigate the number and characteristics of balanced words of length n. Subsequently, Yasutomi [**?**], Kamae and Takahashi [**?**] and Akiyama [**?**] discussed in geometric mapping of balanced words. The unit interval to be divided (vertical axis) in our unit interval integral partition corresponds to the intercept of the Sturmian word in their mapping, whereas the mapping of each order is identical.

3. The three distance theorem

We state the three distance theorem by extending it to include rational values of α.

Theorem 3.1 (Three distance theorem)**.** *Let $0 < \alpha < 1$ be a real number and n a positive integer. The points $\{i\,\alpha\}$, for $0 \le i \le n$, partition the unit interval into at most $n+1$ subintervals, the lengths of which take at most three values; if there are three, one is the sum of the other two.*

More precisely, if $\alpha \in F_n$, then the subintervals of the unit interval divided by $\{i\,\alpha\}$, for $0 \le i \le n$, all have the same length. Otherwise, let $\frac{a}{b} < \frac{c}{d}$ be the Farey pair surrounding α in F_n. Then, the unit interval is divided by the points $\{i\,\alpha\}$, for $0 \le i \le n$, into $n+1$ intervals which satisfy:

- *of these, $n+1-b$ intervals have length $b\,\alpha - a$,*
- *of these, $n+1-d$ intervals have length $c - d\,\alpha$,*
- *of these, $b+d-(n+1)$ intervals have length $(b-d)\,\alpha + c - a$.*

The last case occurs when $b+d > n+1$.

Proof. Each line segment inside U_n can generally be expressed as $h = i\,\alpha - j \quad (i = 1, \dots, n,\ 0 \le j < i,\ 0 < h < 1)$. If two such segments intersect, their α-coordinate is a rational number with denominator less than n, hence in F_{n-1}.

Suppose first that $\alpha \in F_n$. Then the subintervals of the unit interval divided by $h = \{i\,\alpha\}$, for $i = 1, \dots, n$, all have the same length. Indeed, if $\alpha = \frac{p}{q} \in F_n$, then the indeterminate equation $p\,i - q\,j = k$ $(0 < p < q \le n,\ k = 1, \dots, q-1)$ has solutions for each k in the range $0 < i \le n,\ 0 \le j < i$, so the intersections of the segments $h = \{i\,\alpha\}$ with the vertical line $\alpha = \frac{p}{q}$ yield

$$\left\{ \left(\frac{p}{q}, \frac{k}{q} \right) \,\middle|\, k = 1, \dots, q-1 \right\},$$

which correspond to evenly spaced divisions of the unit interval.

Now suppose $\alpha \notin F_n$, and let $\frac{a}{b} < \frac{c}{d}$ be the Farey pair surrounding α in F_n. Define $m = b + d - 1$. Then for every r with $\max\{b, d\} \le r \le m$, this pair remains the Farey pair surrounding α in F_r.

As r increases from $\max\{b, d\}$ to m, we consider how the subdivision of the region U_r evolves. Focus on the vertical strip $\left(\frac{a}{b}, \frac{c}{d}\right) \times (0, 1)$, which corresponds to the Farey interval surrounding α. Within this strip, the Farey pair $\frac{a}{b} < \alpha < \frac{c}{d}$ remains

unchanged for all $r \le m$, so no new vertical boundaries are introduced during this range. Instead, as r increases by one at each step, one of the existing trapezoids within the strip is split into two triangles, since the newly added segment joins existing vertices on the two boundary lines without crossing any existing segment.

In the final subdivision U_m, there are $m+1 = b+d$ regions in the strip. Since the left and right boundaries are divided into b and d equal parts, respectively, and each region has a side on at least one of them, all these regions are triangles: exactly b are based at the left boundary $\left\{\frac{a}{b}\right\} \times (0,1)$ and d at the right boundary $\left\{\frac{c}{d}\right\} \times (0,1)$.

Thus, for general $n \le m$, the number of triangles based at $\left\{\frac{a}{b}\right\} \times (0,1)$ is

$$b - (m-n) = (m+1-d) - (m-n) = n+1-d,$$

and similarly, the number of triangles based at $\left\{\frac{c}{d}\right\} \times (0,1)$ is

$$d - (m-n) = (m+1-b) - (m-n) = n+1-b.$$

The number of remaining trapezoids is

$$m - n = (b+d-1) - n = b+d-(n+1).$$

These regions correspond exactly to subintervals of the unit interval determined by $\{i\,\alpha\}$, and the three lengths appearing in the theorem are derived from the geometry of these triangles and trapezoids. Indeed, since $bc - ad = 1$, the vertical cross-section at α of a triangle based at the left boundary has length $c - d\,\alpha$, that of a triangle based at the right boundary has length $b\,\alpha - a$, and that of a trapezoid is their sum. In particular, the trapezoid case arises precisely when $b+d > n+1$, yielding a third interval length which is the sum of the other two.

Finally, when α equals the mediant $\dfrac{a+c}{b+d}$ of the Farey pair $\dfrac{a}{b} < \dfrac{c}{d}$, the lengths of the subintervals corresponding to the two triangle types coincide. In this case, the number of distinct interval lengths is reduced by one: either from three to two (if trapezoids are present), or from two to one (if no trapezoids appear).

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