

DYNAMIC STUDY AND SEMI-ANALYTICAL SOLUTIONS OF FRACTIONAL-ORDER KELLER–SEGEL COUPLED SYSTEM

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ABSTRACT. In this paper, we studies a fractional-order keller–segel coupled system focusing on their dynamic behavior and nonlinear wave structures. The caputo–fabrizio fractional derivative is used. This derivative has a non-singular kernel. It is suitable for real physical problems. The homotopy perturbation transform method (HPTM) is applied. The method provides an efficient semi-analytical solution. It avoids linearization and discretization. The solution is obtained in series form. The convergence of the solution is smooth and rapid. The dynamic behavior of the system is analyzed. The effect of different fractional orders α is examined. Two-dimensional 2D graphs are plotted for various $\alpha \in (0,1]$ values. Three-dimensional 3D surfaces are also presented. These graphs show the evolution of the solution clearly. Nonlinear wave structures are observed. The amplitude and shape of waves change with α . The results show that fractional order plays a key role. The proposed method is simple and accurate. It can be used for many nonlinear fractional models.

1 Introduction

In recent decades, fractional calculus has drawn a lot of interest. It expands the notion of classical integration and differentiation to non-integer orders. A strong framework for simulating real-world occurrences is offered by this addition. Memory and inherited characteristics are present in many natural and artificial systems. These properties cannot be accurately described using classical integer-order models. Fractional differential equations offer a more realistic representation of such systems [1], [2], [3], [4].

Time-fractional models are particularly important in applied sciences. They integrate temporal memory effects into the dynamics of the system. Because of this, they can be used to explain phenomena including heat conduction, anomalous diffusion, viscoelastic behavior, and biological dynamics. In these systems, the current state depends on both past and present conditions. This nonlocal property is a key advantage of fractional operators [5], [6].

Numerous fields of science and engineering deal with coupled nonlinear systems. Two or more interdependent variables make up these systems. Complex dynamic behavior results from the

interaction of various variables. Examples include fluid–structure interactions, chemical reaction systems, optical fibers, and ecological models. The system displays interesting dynamics when coupling and nonlinearity are combined. These consist of stability transitions, pattern generation, and wave propagation [7], [8], [9].

The capacity to simulate linked nonlinear systems is further improved by incorporating fractional derivatives. Both interaction effects and memory behavior can be captured by time-fractional coupled nonlinear systems. They so offer a more profound understanding of actual physical processes. But the analysis is also more difficult because of this increased complexity. The derivation of exact solutions is frequently impeded by the inclusion of fractional operators and nonlinear factors [10], [11], [12].

Various definitions of fractional derivatives have been introduced in the literature. Classical definitions include the Riemann–Liouville and Caputo derivatives. Although these definitions are widely used, they involve singular kernels. This can create difficulties in both theoretical analysis and numerical implementation. To overcome these limitations, newer definitions have been proposed [13], [14], [15].

One such definition is the Caputo–Fabrizio fractional derivative. This operator is characterized by a non-singular exponential kernel. It avoids the singularity present in classical definitions. This makes it more suitable for modeling real physical systems. The Caputo–Fabrizio derivative also provides a better description of processes with fading memory. Due to these advantages, it has been widely applied in recent studies of fractional models [16], [17], [18]. A key mathematical model in chemotaxis, the nonlinear Keller–Segel coupled system describes how biological organisms, such as bacteria or amoebae, move in response to chemical cues in their surroundings. The model, which was first put forth by Lee Segel and Evelyn Fox Keller, depicts the relationship between chemoattractant concentration and cell density. In its nonlinear form, the system is made up of linked partial differential equations in which the chemical concentration varies as a result of production, degradation, and diffusion, while the cell population density changes as a result of diffusion and directed movement (chemotaxis). Terms that indicate aggregation effects, signal sensitivity, or density-dependent responses are the source of the nonlinearity. Pattern creation, biological aggregation, blow-up events, and spatial-temporal dynamics in biological systems are all extensively studied using this paradigm. Additionally, it provides a crucial framework for the analysis of Dynamic behavior, numerical simulation methods, and nonlinear PDEs in applied mathematics.

In this paper we consider the fractional Non-Linear Keller–Segel Coupled System can be written as [23]

$$\begin{cases} {}^{CF}D_{\tau}^{\alpha}\Phi(\chi, \tau) = A \frac{\partial^2 \Phi(\chi, \tau)}{\partial \chi^2} - \frac{\partial}{\partial \chi} \left(\Phi(\chi, \tau) \frac{\partial \lambda(\rho)}{\partial \chi} \right), \\ {}^{CF}D_{\tau}^{\alpha}\Psi(\chi, \tau) = \zeta \frac{\partial^2 \Psi(\chi, \tau)}{\partial \chi^2} - \eta \Phi(\chi, \tau) - \mu \Psi(\chi, \tau) \end{cases} \quad (1.1)$$

and the initial conditions

$$\Phi(\chi, 0) = \Phi_0(\chi), \quad \Psi(\chi, 0) = \rho_0(\chi), \quad \chi \in I, \quad (1.2)$$

where $\Phi(\chi, \tau)$ denotes the density of cells, $\Psi(\chi, \tau)$ represents the concentration of the chemoattractant, χ is the spatial variable, and τ is time. The parameters A and ζ are the diffusion coefficients of cells and chemical substance, respectively, while η and μ denote the production and

degradation rates of the chemical. The term $-\frac{\partial}{\partial \chi}(\Phi \frac{\partial \Psi}{\partial \chi})$ describes the chemotactic movement of cells toward higher chemical concentration. This coupled nonlinear system effectively captures the interaction between diffusion and chemotaxis mechanisms, leading to complex spatiotemporal dynamics such as aggregation, pattern formation, and possible blow-up behavior.

The remainder of this manuscript is organized as follows: Section 1 provides the introduction to the problem, Section 2 presents Preliminaries, Section 3 presents the mathematical formulation of the nonlinear Keller–Segel coupled system, Section 4 describes the applied methodology for solving the model, Section 5 discusses the obtained results and their interpretation, Section 6 highlights the key findings and implications, and Section 6 concludes the study with future research directions.

2 Preliminaries

In this part of the fundamental notions of fractional derivative and integrals are talked about.

Definition 2.1 The Caputo derivative of $\Phi(t)$ is defined [9].

$${}_a^C D_\tau^\alpha \Phi(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_a^\tau (\tau - \xi)^{\alpha-1} \frac{d^n}{d\tau^n} \Phi(\xi) d\xi. \quad (2.1)$$

Definition 2.2 The Caputo–Fabrizio fractional of $\Phi(t)$ is defined [9].

$${}_0^{CF} D_\tau^\alpha \Phi(\tau) = \frac{\Lambda(\alpha)}{1-\alpha} \int_0^\tau \Phi'(\chi) \exp \left[-\alpha \frac{(\tau-\chi)}{1-\alpha} \right] d\chi. \quad (2.2)$$

This function is a normalization function $\Lambda(\alpha)$ which depend on α with the conditions $\Lambda(0) = \Lambda(1) = 1$.

Definition 2.3 The CFD for integer order is defined as [9].

$${}_0^{CF} D_\tau^\alpha \Phi(\tau) = \frac{2(1-\alpha)}{\Lambda(\alpha)(2-\alpha)} \Phi(\tau) + \frac{2\alpha}{\Lambda(\alpha)(2-\alpha)} \int_0^\tau \Phi(\chi) d\chi. \quad (2.3)$$

Definition 2.4 The Laplace transform (LT) of the CFD is expressed by [9].

$$\begin{aligned} \mathcal{L} \left[{}_0^{CF} D_\tau^{(k)} \Phi(\tau) \right] (w) &= \frac{1}{1-\alpha} \mathcal{L}[\Phi^k(\tau)] \mathcal{L} \left[\exp \left(\frac{-\alpha}{(1-\alpha)} \tau \right) \right] \\ &= \frac{w^k \mathcal{L}[\Phi(\tau)] - w^{k-1} \Phi(0) - w^{k-2} \Phi'(0) \dots - \Phi^{k-1}(0)}{w + \alpha(1-w)}. \end{aligned} \quad (2.4)$$

In this case, we have

$$\begin{aligned} \mathcal{L} \left[{}_0^{CF} D_\tau^{(k+\alpha)} \Phi(\tau) \right] (w) &= \frac{\mathcal{L}\Phi(\tau)}{w + \alpha(1-w)}, \quad k = 0, \\ \mathcal{L} \left[{}_0^{CF} D_\tau^{(k+\alpha)} \Phi(\tau) \right] (w) &= \frac{w \mathcal{L}\Phi(\tau) - \Phi(0)}{w + \alpha(1-w)}, \quad k = 1. \end{aligned}$$

3 Methodology of the Proposed Technique:

Let's look at the following general form of a nonlinear fractional differential equation(NFDEs).

$${}_0^{CF} D_t^\alpha \Phi(\chi, \tau) + \ell \Phi(\chi, \tau) + \aleph \Phi(\chi, \tau) = \Xi(\chi, \tau), \quad (3.1)$$

such that

$$\Phi(\chi, 0) = f(\chi). \quad (3.2)$$

Applying the LT to Eq. (3.1), It follows that:

$$\mathcal{L}[\Phi(\chi, \tau)] = f(\chi) - \left(\frac{w+\alpha(1-w)}{w^{n+1}} \right) \mathcal{L}[\ell \Phi(\chi, \tau) + \aleph \Phi(\chi, \tau)]. \quad (3.3)$$

Taking the inverse Laplace transformation Eq.(3.3), we obtain the following result.

$$\Phi(\chi, \tau) = f(\chi) - \mathcal{L}^{-1} \left[\left(\frac{w+\alpha(1-w)}{w^{n+1}} \right) \mathcal{L}[\ell \Phi(\chi, \tau) + \aleph \Phi(\chi, \tau)] \right]. \quad (3.4)$$

Applying the FHPTM to receive the solutions of Eq. (3.4), we get .

$$\Phi(\chi, \tau) = \sum_{k=0}^{\infty} p^k \Phi_k(\chi, \tau). \quad (3.5)$$

The decomposition of the nonlinear terms is carried out using homotopy polynomials as follows

$$\aleph \Phi(\chi, \tau) = \sum_{k=0}^{\infty} p^k H_k(\chi, \tau). \quad (3.6)$$

The polynomial $H_n(\chi, \tau)$ are define as ([11]).

$$H_k(\Phi_0, \Phi_1, \Phi_2, \dots \Phi_k) = \frac{1}{k!} \frac{\partial^k}{\partial p^k} \left[\left(\sum_{k=0}^{\infty} p^k \Phi_k \right) \right]_{p=0}, \quad k = 0, 1, 2, \dots; \quad (3.7)$$

Putting all these values in Equation (3.4), we obtain

$$\sum_{k=0}^{\infty} p^k \Phi_k(\chi, \tau) = f(\chi) - p \mathcal{L}^{-1} \left[\left(\frac{w+\alpha(1-w)}{w^{k+1}} \right) \mathcal{L}[\beta \sum_{k=0}^{\infty} p^k \Phi_k(\chi, \tau) + \sum_{k=0}^{\infty} p^k H_k] \right]. \quad (3.8)$$

Comparing the coefficients of p^0, p^1, p^2, p^3 and p^4 we get

$$\begin{aligned} \Phi_0(\chi, \tau) &= f(\chi), \\ \Phi_1(\chi, \tau) &= -\mathcal{L}^{-1} \left[\left(\frac{w+\alpha(1-w)}{w^{k+1}} \right) \mathcal{L}[\beta \Phi_0(\chi, \tau) + H_0(\Phi)] \right], \\ \Phi_2(\chi, \tau) &= -\mathcal{L}^{-1} \left[\left(\frac{w+\alpha(1-w)}{w^{k+1}} \right) \mathcal{L}[\beta \Phi_1(\chi, \tau) + H_1(\Phi)] \right], \\ \Phi_3(\chi, \tau) &= -\mathcal{L}^{-1} \left[\left(\frac{w+\alpha(1-w)}{w^{k+1}} \right) \mathcal{L}[\beta \Phi_2(\chi, \tau) + H_2(\Phi)] \right], \\ &\vdots \\ \Phi_{k+1}(\chi, \tau) &= -\mathcal{L}^{-1} \left[\left(\frac{w+\alpha(1-w)}{w^{k+1}} \right) \mathcal{L}[\beta \Phi_k(\chi, \tau) + H_k(\Phi)] \right]. \end{aligned}$$

4 Existence and Uniqueness of the Solution

Theorem 1. *In the region $[0, G] \times S$, where G is finite, the fractional system (1) possesses a unique solution, where*

$$S = \{(\Phi, \Psi) \in \mathbb{R}^2 : \|\Phi\| < \epsilon_1, \|\Psi\| < \epsilon_2\}. \quad (4.1)$$

Proof. Let $V(\tau) = (\Phi(\chi, \tau), \Psi(\chi, \tau))$, $\tilde{V}(\tau) = (\Phi_1(\chi, \tau), \Psi_1(\chi, \tau))$. and $D(\tau, V) = (D_1(\tau, V), D_2(\tau, V))$, where

$$\begin{cases} D_1(\tau, V) = A \frac{\partial^2 \Phi}{\partial \chi^2} - \frac{\partial}{\partial \chi} \left(\Phi \frac{\partial \lambda(\rho)}{\partial \chi} \right), \\ D_2(\tau, C) = \zeta \frac{\partial^2 \Psi}{\partial \chi^2} - \eta \Phi - \mu \Psi. \end{cases} \quad (4.2)$$

Here, $D(t, V)$ is defined on $[0, G] \times S$. Let $E = \sup_S \|D(t, V)\|$ and consider the norm as $\|V(t)\| = \sup_{t \in [0, G]} |V(t)|$. We will demonstrate the existence of a constant $\Lambda > 0$ such that

$$\|D(t, V) - D(t, \tilde{V})\| \leq \Lambda \|V - \tilde{V}\|. \quad (4.3)$$

Consider

$$\|D(t, V) - D(t, \tilde{V})\| = \left\| A \frac{\partial^2 \Phi}{\partial \chi^2} - \frac{\partial}{\partial \chi} \left(\Phi \frac{\partial T^2}{\partial \chi} \right) - A \frac{\partial^2 \Phi_1}{\partial \chi^2} + \frac{\partial}{\partial \chi} \left(\Phi_1 \frac{\partial T^2}{\partial \chi} \right) \right\| \quad (4.4)$$

$$+ \left\| \zeta \frac{\partial^2 \Psi}{\partial \chi^2} + \eta \Phi - \mu \Psi - \zeta \frac{\partial^2 \Psi_1}{\partial \chi^2} - \eta \Phi + \mu \Psi_1 \right\| \quad (4.5)$$

$$\begin{aligned} &\leq (A\theta^2 - B\theta + B) \|\Phi - \Phi_1\| + (\zeta\theta^2 - \mu) \|\Psi - \Psi_1\| \\ &\leq \Lambda_1 \|\Phi - \Phi_1\| + \Lambda_2 \|\Psi - \Psi_1\|, \end{aligned} \quad (4.6)$$

where

$$\Lambda_1 = A\theta^2 - B\theta + B, \quad \Lambda_2 = \zeta\theta^2 - \mu,$$

Assume $\Lambda = \max\{\Lambda_1, \Lambda_2\}$. Therefore, we can write

$$\|D(t, V) - D(t, \tilde{V})\| \leq \Lambda \|V - \tilde{V}\|. \quad (4.7)$$

We will construct Picard's operator using the fractional integral as:

$$\varpi V = V(0) + I^\rho D(t, V) \quad (4.8)$$

To construct the contraction map, we map the operator to a non-empty full metric. Suppose

$$\|V - V(0)\| \leq \hat{l} \quad (4.9)$$

Normalizing both sides of Equation (4.8) yields

$$\| \varpi V - V(0) \| \leq \| D(t, V) \| I^\rho \leq E \frac{G^\rho}{\Gamma(\rho+1)} < \hat{l}. \quad (4.10)$$

The inequality holds when

$$\frac{G^\rho}{\Gamma(\rho+1)} < \frac{\hat{l}}{E}.$$

Since a condition has been added to turn ϖ into a contraction map, we begin with

$$\begin{aligned} \| \varpi V - \varpi \tilde{V} \| &\leq \| I^\rho (D(t, V) - D(t, \tilde{V})) \| \\ &\leq I^\rho \| D(t, V) - D(t, \tilde{V}) \| \\ &\leq \| D(t, V) - D(t, \tilde{V}) \| I^\rho(1) \\ &\leq \frac{\tilde{G}^\rho}{\Gamma(\rho+1)} \Lambda \| V - \tilde{V} \|. \end{aligned} \quad (4.11)$$

We infer that ϖ is a contraction from Equation (4.11) if

$$\frac{G^\rho}{\Gamma(\rho+1)} \leq \frac{1}{\Lambda} \quad (4.12)$$

The resulting ϖ is a contraction map from the above condition. We apply Banach's Fixed-Point Theorem to guarantee that Δ has a single fixed point. Furthermore, the System (1.1) has a unique solution when

$$\frac{G^\rho}{\Gamma(\rho+1)} \leq \min \left(\frac{\hat{l}}{E}, \frac{1}{\Lambda} \right). \quad (4.13)$$

Hence proof holds for the Caputo-Fabrizio fractional derivative

5 Semi-analytical solutions

Example 1. Consider the Keller–Segel equation with the sensitivity function $\chi(\rho) = 1$. Then, the chemo–tactic term

$$\frac{\partial}{\partial \chi} \left(\Phi(\chi, \tau) \frac{\partial \chi(\rho)}{\partial \chi} \right) = 0, \quad (5.1)$$

$${}^{CF}D_\tau^\alpha \Phi(\chi, \tau) = A \frac{\partial^2 \Phi(\chi, \tau)}{\partial \chi^2}, \quad (5.2)$$

$${}^{CF}D_\tau^\alpha \psi(\chi, \tau) = \zeta \frac{\partial^2 \psi(\chi, \tau)}{\partial \chi^2} + \eta \Phi(\chi, \tau) - \mu \psi(\chi, \tau), \quad (5.3)$$

subject to the initial conditions

$$\Phi(\chi, 0) = p_1 e^{-\chi^2}, \quad \psi(\chi, 0) = q_1 e^{-\chi^2}, \quad \chi \in I.$$

By HTPM, we obtain the following Eq. (3.2)

$$\sum_{n=0}^{\infty} p^n \Phi_n(\chi, \tau) - \Phi(\chi, 0) = pL^{-1} \left[\left(\frac{s+\alpha(1-s)}{s} \right) \mathcal{L} \left(A \frac{\partial^2}{\partial \chi^2} (\sum_{n=0}^{\infty} p^n \Phi_n(\chi, \tau)) \right) \right], \quad (5.4)$$

$$\begin{aligned} \sum_{n=0}^{\infty} p^n \rho_n(\chi, \tau) &= \rho(\chi, 0) \\ &+ pL^{-1} \left[\left(\frac{s+\alpha(1-s)}{s} \right) \mathcal{L} \left[\zeta \frac{\partial^2}{\partial \chi^2} (\sum_{n=0}^{\infty} p^n \Psi_n(\chi, \tau)) + \eta \sum_{n=0}^{\infty} p^n \Phi_n(\chi, \tau) - \mu \sum_{n=0}^{\infty} p^n \Psi_n(\chi, \tau) \right] \right]. \end{aligned} \quad (5.5)$$

Firmly facing the above conditions, we get

$$\begin{aligned} \Phi_0(\chi, \tau) &= p_1 e^{-\chi^2}, \\ \psi_0(\chi, \tau) &= q_1 e^{-\chi^2}, \\ \Phi_1(\chi, \tau) &= A(1 - \alpha + t\alpha)(-2e^{-\chi^2} + 4\chi^2 e^{-\chi^2})p_1 \\ \psi_1(\chi, \tau) &= (1 - \alpha + t\alpha)[e^{-\chi^2} \eta p_1 - e^{-\chi^2} \mu q_1 + \zeta(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2)q_1], \\ \Phi_2(\chi, \tau) &= A^2 \left(1 - 2\alpha + \alpha^2 + \frac{t^2 \alpha^2}{2} - 2t(-\alpha + \alpha^2) \right) \\ &\quad \times \left(8e^{-\chi^2} - 32\chi^2 e^{-\chi^2} - 2(-2e^{-\chi^2} + 4\chi^2 e^{-\chi^2}) + 4\chi^2(-2e^{-\chi^2} + 4\chi^2 e^{-\chi^2}) \right) p_1 \\ \psi_2(\chi, \tau) &= \frac{1}{2} e^{-\chi^2} t^2 \alpha^2 (-2A\eta p_1 - 2\zeta \eta p_1 - \eta \mu p_1 + 4A\eta \chi^2 p_1 + 4\zeta \eta \chi^2 p_1 + 12\zeta^2 q_1 + 4\zeta \mu q_1 + \mu^2 q_1 \\ &\quad - 48\zeta^2 \chi^2 q_1 - 8\zeta \mu \chi^2 q_1 + 16\zeta^2 \chi^4 q_1) \\ &\quad + e^{-\chi^2} (-2A\eta p_1 + 4A\alpha \eta p_1 - 2A\alpha^2 \eta p_1 - 2\zeta \eta p_1 + 4\alpha \zeta \eta p_1 - 2\alpha^2 \zeta \eta p_1 \\ &\quad - \eta \mu p_1 + 2\alpha \eta \mu p_1 - \alpha^2 \eta \mu p_1 + 4A\eta \chi^2 p_1 - 8A\alpha \eta \chi^2 p_1 + 4A\alpha^2 \eta \chi^2 p_1 \\ &\quad + 4\zeta \eta \chi^2 p_1 - 8\alpha \zeta \eta \chi^2 p_1 + 4\alpha^2 \zeta \eta \chi^2 p_1 + 12\zeta^2 q_1 - 24\alpha \zeta^2 q_1 + 12\alpha^2 \zeta^2 q_1 + 4\zeta \mu q_1 - 8\alpha \zeta \mu q_1 \\ &\quad - 8\zeta \mu \chi^2 q_1 + 16\alpha \zeta \mu \chi^2 q_1 - 8\alpha^2 \zeta \mu \chi^2 q_1 + 16\zeta^2 \chi^4 q_1 - 32\alpha \zeta^2 \chi^4 q_1 + 16\alpha^2 \zeta^2 \chi^4 q_1) \\ &\quad - 2e^{-\chi^2} t(2A\alpha \eta p_1 - 2A\alpha^2 \eta p_1 + 2\alpha \zeta \eta p_1 - 2\alpha^2 \zeta \eta p_1 + \alpha \eta \mu p_1 - \alpha^2 \eta \mu p_1 - 4A\alpha \eta \chi^2 p_1 \\ &\quad + 4A\alpha^2 \eta \chi^2 p_1 - 4\alpha \zeta \eta \chi^2 p_1 + 4\alpha^2 \zeta \eta \chi^2 p_1 - 12\alpha \zeta^2 q_1 + 12\alpha^2 \zeta^2 q_1 - 4\alpha \zeta \mu q_1 + 4\alpha^2 \zeta \mu q_1 \\ &\quad - \alpha \mu^2 q_1 + \alpha^2 \mu^2 q_1 + 48\alpha \zeta^2 \chi^2 q_1 - 48\alpha^2 \zeta^2 \chi^2 q_1 + 8\alpha \zeta \mu \chi^2 q_1 - 8\alpha^2 \zeta \mu \chi^2 q_1 \\ &\quad - 16\alpha \zeta^2 \chi^4 q_1 + 16\alpha^2 \zeta^2 \chi^4 q_1) \\ \Phi_3(\chi, \tau) &= A^3 \left(1 - 3\alpha + 3\alpha^2 - \alpha^3 + \frac{t^3 \alpha^3}{6} + 3t(\alpha - 2\alpha^2 + \alpha^3) - \frac{3}{2} t^2(-\alpha^2 + \alpha^3) \right) \\ &\quad - 32\chi^2(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2) + 16\chi(12e^{-\chi^2} \chi - 8e^{-\chi^2} \chi^3) \\ &\quad - 2(8e^{-\chi^2} - 32e^{-\chi^2} \chi^2 - 2(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2) \\ &\quad + 4\chi^2(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2)) \\ &\quad + 4\chi^2(8e^{-\chi^2} - 32e^{-\chi^2} \chi^2 - 2(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2) \\ &\quad + 4\chi^2(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2)) p_1. \\ &\vdots \end{aligned}$$

Hence, the truncated approximate analytical solution becomes

$$\Phi_{\text{app}} = \sum_{n=0}^3 \Phi_n, \quad \psi_{\text{app}} = \sum_{n=0}^2 \psi_n \quad (5.6)$$

Example 2. Consider the following Keller–Segel equation with the sensitivity function $\lambda(\psi) = \psi$, $A = \beta$, with the chemotactic term

$$\begin{aligned} \frac{\partial}{\partial \chi} \left(\Phi(\chi, \tau) \frac{\partial \lambda(\rho)}{\partial \chi} \right) &= \frac{\partial \Phi(\chi, \tau)}{\partial \chi} \frac{\partial \psi(\chi, \tau)}{\partial \chi} + \Phi(\chi, \tau) \frac{\partial^2 \psi(\chi, \tau)}{\partial \chi^2}. \\ {}^{CF}D_{\tau}^{\alpha} \Phi(\chi, \tau) &= \beta \frac{\partial^2 \Phi(\chi, \tau)}{\partial \chi^2} - \frac{\partial \Phi(\chi, \tau)}{\partial \chi} \frac{\partial \psi(\chi, \tau)}{\partial \chi} + \Phi(\chi, \tau) \frac{\partial^2 \psi(\chi, \tau)}{\partial \chi^2}, \end{aligned} \quad (5.7)$$

$${}^{CF}D_{\tau}^{\alpha} \psi(\chi, \tau) = \zeta \frac{\partial^2 \psi(\chi, \tau)}{\partial \chi^2} + \eta \Phi(\chi, \tau) - \mu \psi(\chi, \tau), \quad (5.8)$$

and subject to the initial conditions

$$\Phi(\chi, 0) = p_1 e^{-\chi^2}, \quad \psi(\chi, 0) = q_1 e^{-\chi^2}, \quad \chi \in I. \quad (3.7)$$

By applying the proposed Homotopy Perturbation Transform Technique (HPTT) to Equations (3.15) to (3.16), subject to the given initial conditions, the solution is obtained as follows:

$$\begin{aligned} \Phi_0(\chi, \tau) &= p_1 e^{-\chi^2}, \\ \psi_0(\chi, \tau) &= q_1 e^{-\chi^2}, \\ \Phi_1(\chi, \tau) &= (1 - \alpha + t\alpha) [\beta(-2e^{-\chi^2} + 4\chi^2 e^{-\chi^2})p_1 + 4\chi^2 e^{-2\chi^2} p_1 q_1 + e^{-\chi^2}(-2e^{-\chi^2} + 4\chi^2 e^{-\chi^2})p_1 q_1], \\ \psi_1(\chi, \tau) &= (1 - \alpha + t\alpha) [e^{-\chi^2} \eta p_1 - e^{-\chi^2} \mu q_1 + \zeta(-2e^{-\chi^2} + 4e^{-\chi^2} \chi^2)q_1] \\ \Phi_2(\chi, \tau) &= 2e^{-3\chi^2} [t^2 \alpha^2 A(\chi) + 2(1 - \alpha)^2 A(\chi) - 4t\alpha(\alpha - 1)A(\chi)] \\ \psi_2(\chi, \tau) &= e^{-2\chi^2} \left[\left(\frac{t^2 \alpha^2}{2} + (1 - \alpha)^2 \right) F(\chi) - 2t\alpha(\alpha - 1)C(\chi) \right] \end{aligned} \quad (5.9)$$

where

$$\begin{aligned} A(\chi) &= 3e^{2\chi^2} \beta^2 p_1 - 12e^{2\chi^2} \beta^2 \chi^2 p_1 + 4e^{2\chi^2} \beta^2 \chi^4 p_1 + e^{\chi^2} \eta p_1^2 (\chi + \chi^2 - 2\chi^3) \\ &\quad + e^{\chi^2} p_1 q_1 (7\beta - 6\zeta\chi - \mu\chi - 58\beta\chi^2 - 6\zeta\chi^2 - \mu\chi^2 + 24\zeta\chi^3 + 2\mu\chi^3 + 40\beta\chi^4 + 4\zeta\chi^4 - 8\zeta\chi^5) \\ &\quad + p_1 q_1^2 (1 - 18\chi^2 + 24\chi^4). \end{aligned}$$

$$\begin{aligned} F(\chi) &= -2e^{\chi^2} \eta (\beta + \zeta) p_1 - e^{\chi^2} \eta \mu p_1 + 4e^{\chi^2} \eta (\beta + \zeta) \chi^2 p_1 \\ &\quad + e^{\chi^2} q_1 (12\zeta^2 + 4\zeta\mu + \mu^2 - 48\zeta^2 \chi^2 - 8\zeta\mu\chi^2 + 16\zeta^2 \chi^4) \\ &\quad - 2\eta p_1 q_1 + 8\eta \chi^2 p_1 q_1 \end{aligned}$$

$$\begin{aligned} G(\chi) &= 2e^{\chi^2} \eta (\beta + \zeta) p_1 + e^{\chi^2} \eta \mu p_1 - 4e^{\chi^2} \eta (\beta + \zeta) \chi^2 p_1 \\ &\quad - e^{\chi^2} q_1 (12\zeta^2 + 4\zeta\mu + \mu^2 - 48\zeta^2 \chi^2 - 8\zeta\mu\chi^2 + 16\zeta^2 \chi^4) \\ &\quad + 2\eta p_1 q_1 - 8\eta \chi^2 p_1 q_1 \end{aligned}$$

Hence, the truncated approximate analytical solution becomes

$$\Phi_{\text{app}} = \sum_{n=0}^2 \Phi_n, \quad \psi_{\text{app}} = \sum_{n=0}^2 \psi_n \quad (5.10)$$

Example 3. Consider the following Keller-Segel equation with the sensitivity function $\lambda(\psi) = \psi^2, A = \beta$, with the chemotactic term

$$\frac{\partial}{\partial \chi} \left(\Phi(\chi, \tau) \frac{\partial \lambda(\rho)}{\partial \chi} \right) = \frac{\partial \Phi(\chi, \tau)}{\partial \chi} \frac{\partial \psi^2(\chi, \tau)}{\partial \chi} + \Phi(\chi, \tau) \frac{\partial^2 \psi^2(\chi, \tau)}{\partial \chi^2}.$$

$${}^{CF}D_{\tau}^{\alpha}\Phi(\chi, \tau) = \beta \frac{\partial^2 \Phi(\chi, \tau)}{\partial \chi^2} - \frac{\partial \Phi(\chi, \tau)}{\partial \chi} \frac{\partial \psi^2(\chi, \tau)}{\partial \chi} + \Phi(\chi, \tau) \frac{\partial^2 \psi^2(\chi, \tau)}{\partial \chi^2}, \quad (5.11)$$

$${}^{CF}D_{\tau}^{\alpha}\psi(\chi, \tau) = \zeta \frac{\partial^2 \psi(\chi, \tau)}{\partial \chi^2} + \eta \Phi(\chi, \tau) - \mu \psi(\chi, \tau), \quad (5.12)$$

and subject to the initial conditions

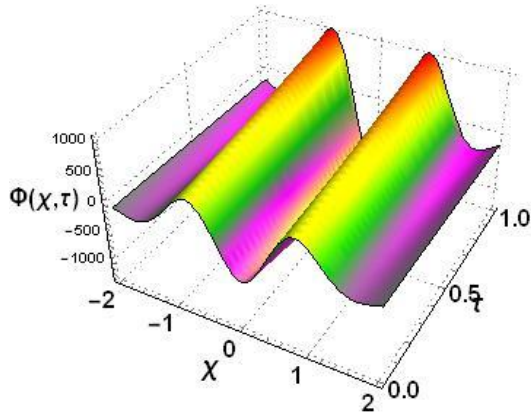
$$\Phi(\chi, 0) = p_1 \sin(\chi), \quad \psi(\chi, 0) = q_1 \sin(\chi), \quad \chi \in I. \quad (3.7)$$

By applying the proposed Homotopy Perturbation Transform Technique (HPTT) to Equations (3.15) to (3.16), subject to the given initial conditions, the solution is obtained as follows:

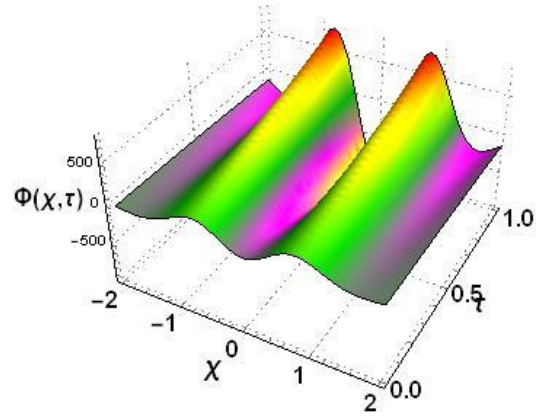
$$\begin{aligned} \Phi_0(\chi, \tau) &= p_1 \sin(\chi), \\ \psi_0(\chi, \tau) &= q_1 \sin(\chi), \\ \Phi_1(\chi, \tau) &= (1 - \alpha + t\alpha)[- \beta p_1 \sin(\chi) - p_1 q_1^2 \cos^3(\chi) - p_1 q_1^2 \sin^3(\chi)], \\ \psi_1(\chi, \tau) &= (1 - \alpha + t\alpha)[\eta p_1 \sin(\chi) - (\zeta + \mu) q_1 \sin(\chi)], \\ \Phi_2(\chi, \tau) &= -p_1 \left(1 - 2\alpha + \alpha^2 + \frac{t^2 \alpha^2}{2} - 2t(-\alpha + \alpha^2) \right) \\ &\quad \times [6q_1^2 \beta \cos^2 \chi \sin \chi + 3q_1^4 \cos^4 \chi \sin \chi \\ &\quad + 6q_1^2 \beta \cos \chi \sin^2 \chi - 2q_1 \cos^3 \chi (-p_1 \eta + q_1 (2\beta + \zeta + \mu) + 2q_1^3 \sin^2 \chi) \\ &\quad - \sin \chi (\beta^2 + 2q_1 (-p_1 \eta + q_1 (2\beta + \zeta + \mu)) \sin^2 \chi + q_1^4 \sin^4 \chi)], \\ \psi_2(\chi, \tau) &= -p_1 q_1^2 \eta \cos^3(\chi) + 2p_1 q_1^2 \alpha \eta \cos^3(\chi) - p_1 q_1^2 \alpha^2 \eta \cos^3(\chi) \\ &\quad + q_1 \zeta^2 \sin(\chi) - 2q_1 \alpha \zeta^2 \sin(\chi) + q_1 \alpha^2 \zeta^2 \sin(\chi) \\ &\quad - p_1 \beta \eta \sin(\chi) + 2p_1 \alpha \beta \eta \sin(\chi) - p_1 \alpha^2 \beta \eta \sin(\chi) \\ &\quad - p_1 \zeta \eta \sin(\chi) + 2p_1 \alpha \zeta \eta \sin(\chi) - p_1 \alpha^2 \zeta \eta \sin(\chi) \\ &\quad + 2q_1 \zeta \mu \sin(\chi) - 4q_1 \alpha \zeta \mu \sin(\chi) + 2q_1 \alpha^2 \zeta \mu \sin(\chi) \\ &\quad - p_1 \eta \mu \sin(\chi) + 2p_1 \alpha \eta \mu \sin(\chi) - p_1 \alpha^2 \eta \mu \sin(\chi) \\ &\quad + q_1 \mu^2 \sin(\chi) - 2q_1 \alpha \mu^2 \sin(\chi) + q_1 \alpha^2 \mu^2 \sin(\chi) \\ &\quad - p_1 q_1^2 \eta \sin^3(\chi) + 2p_1 q_1^2 \alpha \eta \sin^3(\chi) - p_1 q_1^2 \alpha^2 \eta \sin^3(\chi) \\ &\quad - \frac{1}{2} t^2 \alpha^2 (p_1 q_1^2 \eta \cos^3(\chi) - q_1 \zeta^2 \sin(\chi) + p_1 \beta \eta \sin(\chi) + p_1 \zeta \eta \sin(\chi) - 2q_1 \zeta \mu \sin(\chi) \\ &\quad + p_1 \eta \mu \sin(\chi) - q_1 \mu^2 \sin(\chi) + p_1 q_1^2 \eta \sin^3(\chi)) \\ &\quad + 2t(-p_1 q_1^2 \alpha \eta \cos^3(\chi) + p_1 q_1^2 \alpha^2 \eta \cos^3(\chi) + q_1 \alpha \zeta^2 \sin(\chi) - q_1 \alpha^2 \zeta^2 \sin(\chi) \\ &\quad - p_1 \alpha \beta \eta \sin(\chi) + p_1 \alpha^2 \beta \eta \sin(\chi) - p_1 \alpha \zeta \eta \sin(\chi) + p_1 \alpha^2 \zeta \eta \sin(\chi) + 2q_1 \alpha \zeta \mu \sin(\chi) \\ &\quad - 2q_1 \alpha^2 \zeta \mu \sin(\chi) - p_1 \alpha \eta \mu \sin(\chi) + p_1 \alpha^2 \eta \mu \sin(\chi) + q_1 \alpha \mu^2 \sin(\chi) - q_1 \alpha^2 \mu^2 \sin(\chi) \\ &\quad - p_1 q_1^2 \alpha \eta \sin^3(\chi) + p_1 q_1^2 \alpha^2 \eta \sin^3(\chi)). \end{aligned} \quad (5.13)$$

Hence, the truncated approximate analytical solution becomes

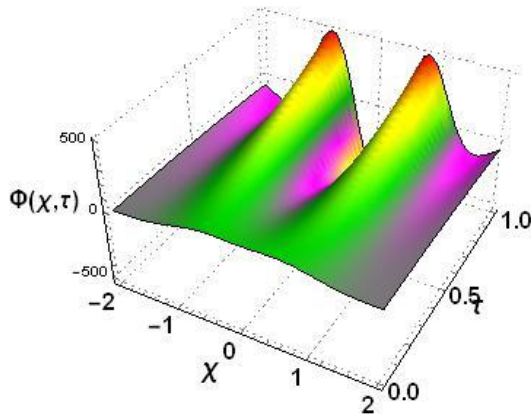
$$\Phi_{\text{app}} = \sum_{n=0}^2 \Phi_n, \quad \psi_{\text{app}} = \sum_{n=0}^2 \psi_n \quad (5.14)$$



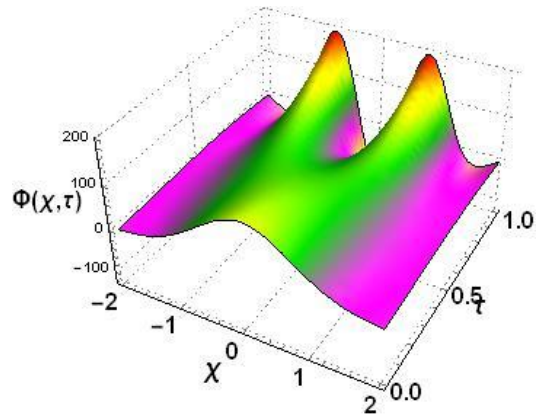
(a) Dynamic Wave Surface $\alpha = 0.25$



(b) Dynamic Wave Surface $\alpha = 0.5$



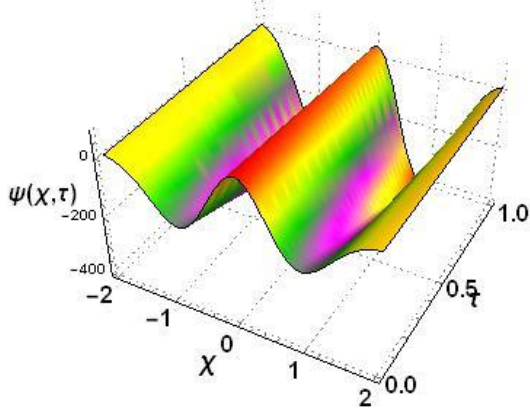
(c) Dynamic Wave Surface $\alpha = 0.75$



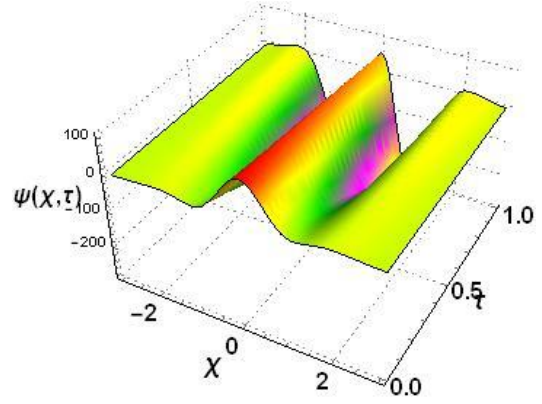
(d) Dynamic Wave Surface $\alpha = 1$

Figure 1. Wave propagation surface of the solution $\Phi(x, \tau)$ illustrating the smooth and stable behavior over the computational domain.

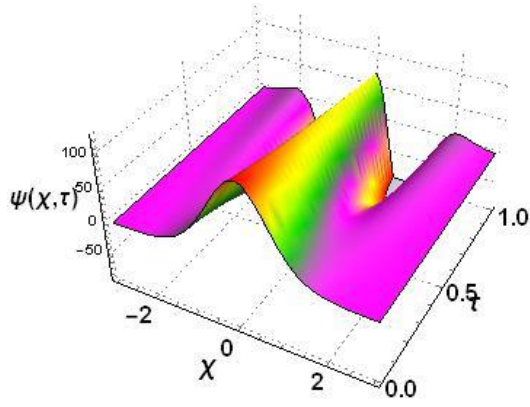
Wave propagation surface of the solution $\Phi(x, \tau)$ illustrating the smooth and stable behavior over the computational domain.



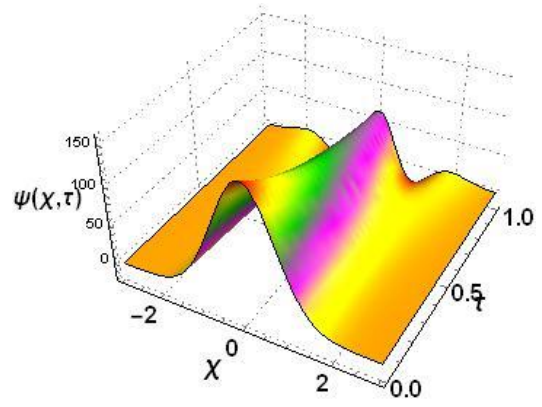
(a) Dynamic Wave Surface $\alpha = 0.25$



(b) Dynamic Wave Surface $\alpha = 0.5$

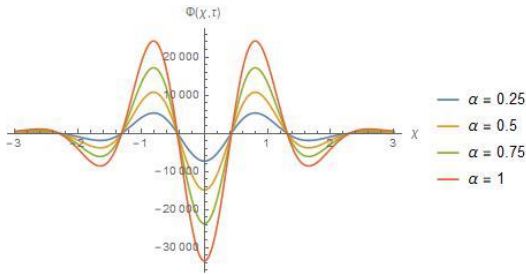


(c) Dynamic Wave Surface $\alpha = 0.75$

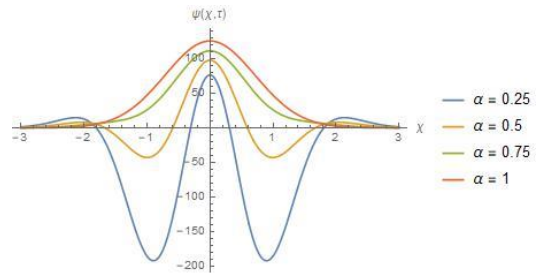


(d) Dynamic Wave Surface $\alpha = 1$

Figure 2. Wave propagation surface of the solution $\psi(x, \tau)$ illustrating the smooth and stable behavior over the computational domain.

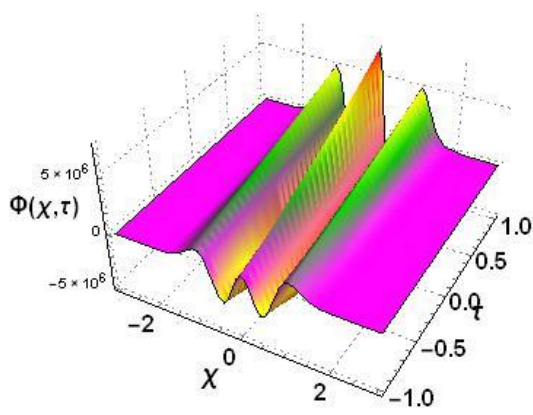


(a) Variation $\Phi(x, \tau)$

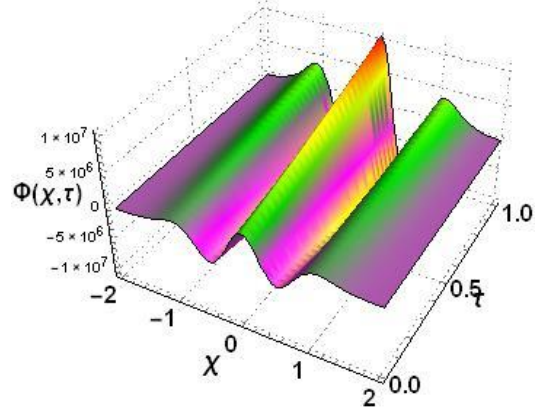


(b) Variation $\psi(x, \tau)$.

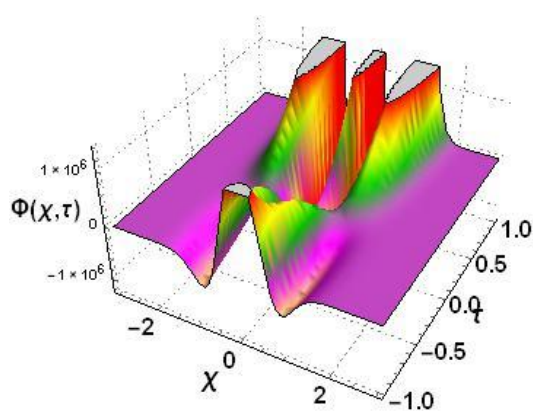
Figure 3. (a), 2D graphical representation of the approximate solution $\Phi(x, \tau)$ obtained by HPTM for different fractional orders α , (b) $\psi(x, \tau)$ approximate solution in 2D view.



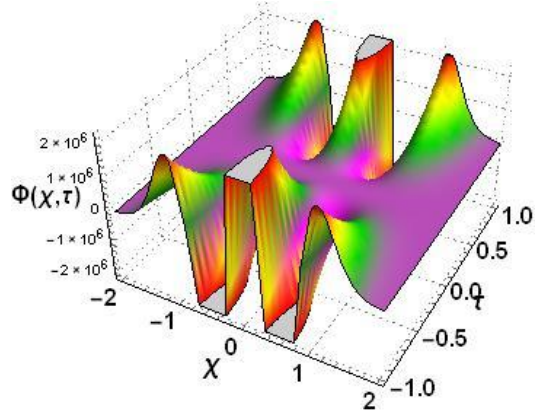
(a) Dynamic Wave Surface $\alpha = 0.25$



(b) Dynamic Wave Surface $\alpha = 0.5$

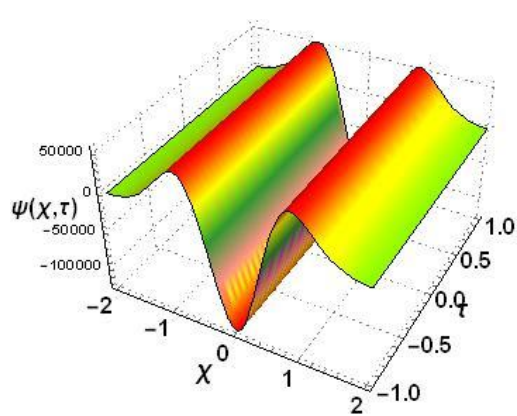


(c) Dynamic Wave Surface $\alpha = 0.75$

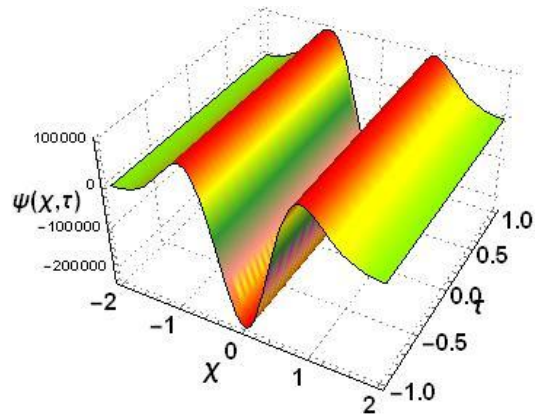


(d) Dynamic Wave Surface $\alpha = 1$

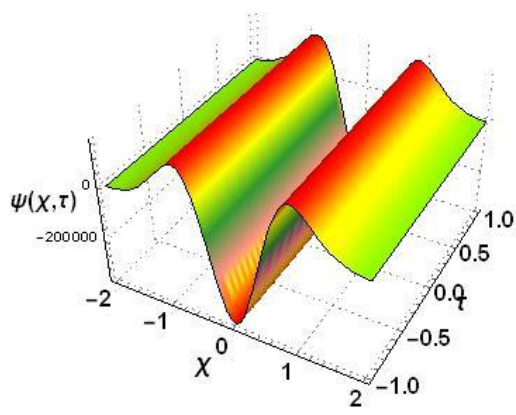
Figure 4. Wave propagation surface of the solution $\Phi(\chi, \tau)$ illustrating the smooth and stable behavior over the computational domain.



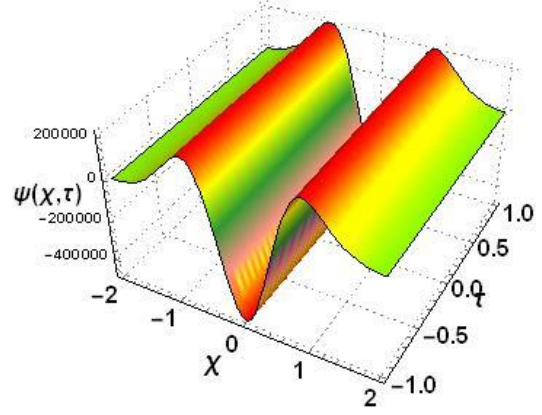
(a) Dynamic Wave Surface $\alpha = 0.25$



(b) Dynamic Wave Surface $\alpha = 0.5$

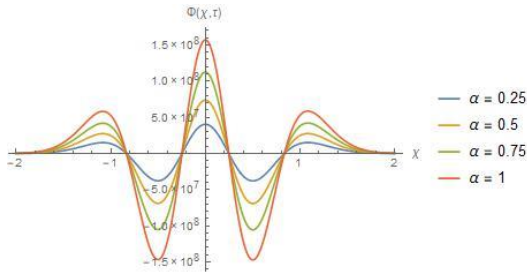


(c) Dynamic Wave Surface $\alpha = 0.75$

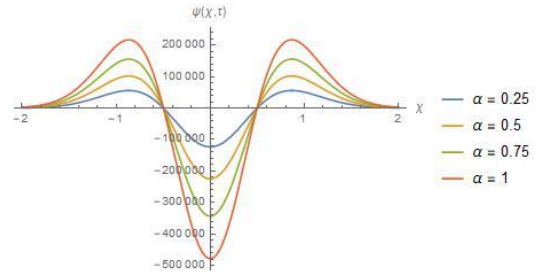


(d) Dynamic Wave Surface $\alpha = 1$

Figure 5. Wave propagation surface of the solution $\psi(\chi, \tau)$ illustrating the smooth and stable behavior over the computational domain.

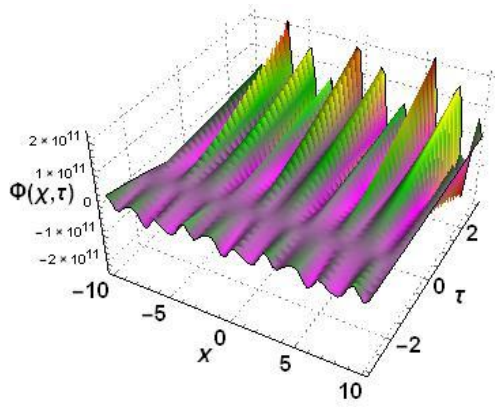


(a) Variation $\Phi(\chi, \tau)$

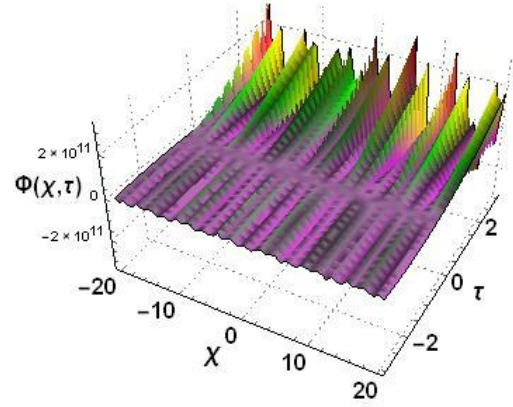


(b) Variation $\psi(\chi, \tau)$.

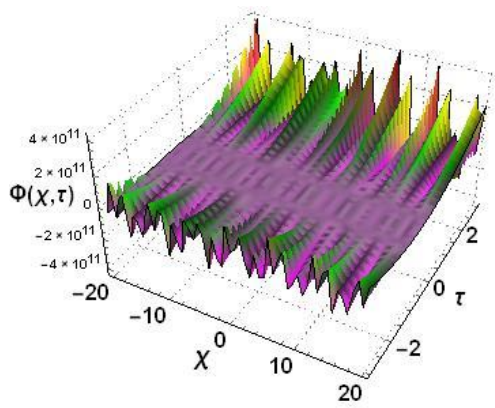
Figure 6. (a), 2D graphical representation of the approximate solution $\Phi(\chi, \tau)$ obtained by HPTM for different fractional orders α , (b) $\psi(\chi, \tau)$ approximate solution in 2D view.



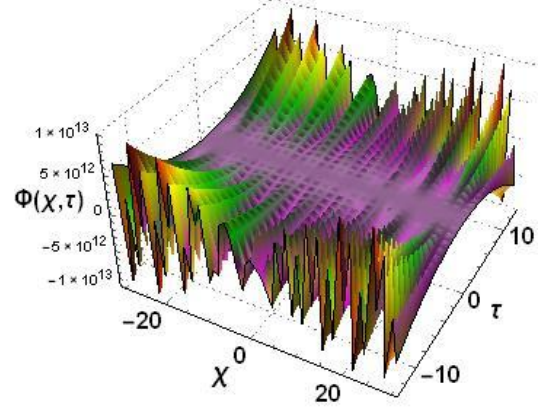
(a) Dynamic Wave Surface $\alpha = 0.25$



(b) Dynamic Wave Surface $\alpha = 0.5$

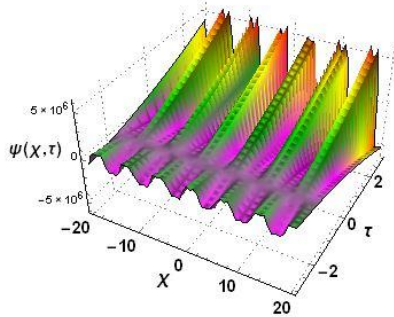


(c) Dynamic Wave Surface $\alpha = 0.75$

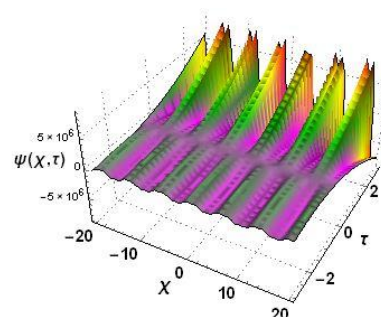


(d) Dynamic Wave Surface $\alpha = 1$

Figure 7. Wave propagation surface of the solution $\Phi(\chi, \tau)$ illustrating the smooth and stable behavior over the computational domain.



(a) Dynamic Wave Surface $\alpha = 0.25$



(b) Dynamic Wave Surface $\alpha = 0.5$

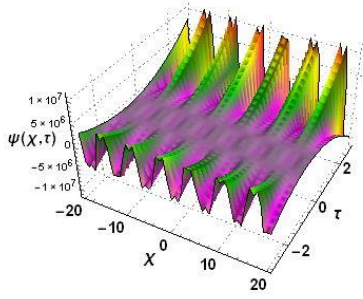
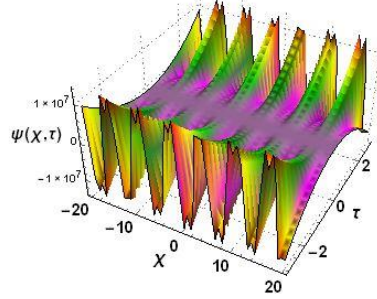
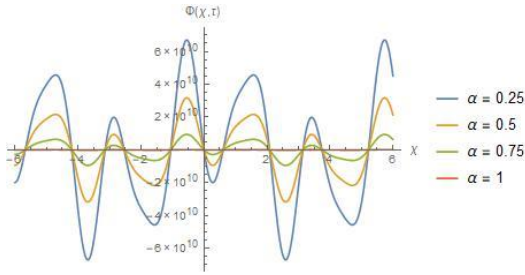
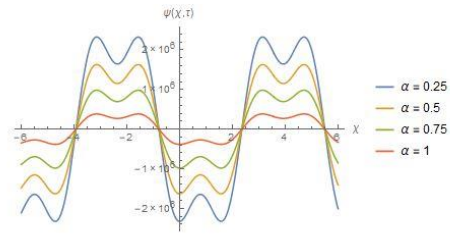

 (c) Dynamic Wave Surface $\alpha = 0.75$

 (d) Dynamic Wave Surface $\alpha = 1$
Figure 8. Wave propagation surface of the solution $\psi(\chi, \tau)$ illustrating the smooth and stable behavior over the computational domain.

 (a) Variation $\Phi(\chi, \tau)$

 (b) Variation $\psi(\chi, \tau)$.

Figure 9. (a), 2D graphical representation of the approximate solution $\Phi(\chi, \tau)$ obtained by HPTM for different fractional orders α , (b) $\psi(\chi, \tau)$ approximate solution in 2D view.

6 Conclusion

In this study, the nonlinear Keller–Segel coupled system has been successfully analyzed using the Homotopy Perturbation Transform Method within the framework of the Caputo–Fabrizio fractional derivative, incorporating fractional analysis to capture the memory-dependent and nonlocal characteristics of the system; the obtained approximate solutions exhibit good convergence and accuracy, while the 2D and 3D graphical representations effectively illustrate the spatiotemporal dynamics, diffusion–chemotaxis interaction, and aggregation behavior, thereby confirming the efficiency and reliability of the proposed approach for analyzing complex nonlinear fractional-order biological models.

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Author contributions

All authors worked equally in this paper.

Data availability

No datasets were generated or analysed during the current study.

Declarations

We confirm that this manuscript is an original work and has not been submitted elsewhere for consideration.

Ethics approval and consent to participate

This is not applicable.

Competing interests

The authors declare no competing interests.

Conflict of interest

Authors declares no conflict of interest.

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