

Periodic Adjacent Couplings for Barycentric Shifts in a Hybrid Sequence Space

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Abstract. This study examines the right barycentric shift on a probability simplex contained in a hybrid weighted sequence space. The underlying hybrid norm is obtained from the intersection of two multiplier components associated with a boundary-anchored Cesaro–difference domain. After a diagonal isometry transfers the simplex to the space of absolutely summable sequences, an adjacent-coupling term is added to the norm. The Lipschitz coefficient of every iterate is then represented by a variational formula over zero-sum coefficient vectors. For periodic coupling profiles, the formula reduces the asymptotic question to finitely many residue classes. We prove that the shift is asymptotically nonexpansive precisely when the periodic profile is identically zero. The same argument shows that an eventually periodic profile has this property precisely when the entire profile is zero: a nonzero periodic tail creates a recurrent boundary defect, whereas a nonzero initial coupling creates a persistent denominator defect after the tail has vanished. Restricting the variational problem to vectors supported on two consecutive periods gives finite-dimensional linear programs and computable certified lower bounds for the iterate defect. For a constant profile, the resulting expression agrees with the exact coefficient. These results separate the geometry of the underlying weighted space from the additional information carried by adjacent couplings.

1. Introduction

The fixed point theory of nonexpansive mappings began with geometric existence theorems such as Kirk’s theorem [1]. Goebel and Kirk introduced asymptotically nonexpansive mappings [2] and subsequently studied mappings whose iterates have a common Lipschitz bound [3]. For nonreflexive spaces these global properties can fail, so one is led to ask which explicitly described closed bounded convex subsets retain a fixed point theorem. Goebel and Kuczumow constructed irregular simplices with this property [4]; Kaczor and Prus later studied related finite-perturbation and finite-minimum configurations in ℓ_1 [5]. Alspach’s example [6] shows at the same time that a weakly compact convex subset of $L^1[0, 1]$ may support a fixed-point-free nonexpansive map.

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Cesaro and generalized difference matrix domains provide a separate source of weighted coordinate spaces. Their Kothe–Toeplitz multiplier duals were computed in several settings; see [7, 8]. The hybrid space used here is the intersection of two explicitly stated multiplier components. The definition is given below from the boundary-anchored Cesaro–difference domain, rather than introduced as an unexplained weighted ℓ_1 space. The diagonal normalization $J : H \rightarrow \ell_1$ is a proof device: all subsets, mappings and norm estimates are defined first and transported only through an explicitly displayed isometry.

We classify periodic and eventually periodic coupling profiles and reduce persistent iterate defects to finitely many residue classes.

The fixed point property of a Banach space quantifies over every nonempty closed bounded convex subset. None of the subset theorems below makes that global assertion. Each conclusion is tied to its displayed probability simplex and to the stated mapping class. This distinction is essential when the results are compared with fixed-point-free shifts or function-space counterexamples.

2. Methods and mathematical framework

Let ω be the space of all scalar sequences, let $q \in \mathbb{N}$, and let $v = (v_n)$ have no zero terms. With the forward-difference convention, set

$$(\Delta_v^q x)_n = \Delta^q(vx)_n = \sum_{j=0}^q (-1)^j \binom{q}{j} v_{n+j} x_{n+j}.$$

Let C denote the Cesaro matrix, $(Cz)_n = n^{-1} \sum_{k=1}^n z_k$, and define the boundary-anchored domain

$$\mathcal{E}_{C,v}^q = \{x \in \omega : C\Delta_v^q x \in \ell_\infty\},$$

equipped with

$$\|x\|_{\mathcal{E}} = \max \left\{ \max_{1 \leq j \leq q} |v_j x_j|, \|C\Delta_v^q x\|_\infty \right\}.$$

Generalized difference domains and their duals are studied in [7, 8]. The boundary block is necessary: the transform vanishes whenever $v_n x_n$ is a polynomial of degree at most $q - 1$. The Kothe–Toeplitz alpha-dual of a sequence space X is

$$X^\alpha = \left\{ a : \sum_{n \geq 1} |a_n x_n| < \infty \text{ for every } x \in X \right\}.$$

For the boundary-anchored domain one obtains the multiplier dual

$$U_1^q(v) = (\mathcal{E}_{C,v}^q)^\alpha = \ell_1(u), \quad u_n = \frac{n^q}{|v_n|}.$$

The complementary Cesaro–difference inversion component is

$$W_1^q(v) = \ell_1(w), \quad w_n = \frac{1}{n^q |v_n|}.$$

We define the hybrid dual and its weight by

$$H = H_1^{q,\#}(v) = U_1^q(v) \cap W_1^q(v) = \ell_1(h), \quad h_n = u_n + w_n = \frac{n^{2q} + 1}{n^q |v_n|},$$

with the sum norm

$$\|a\|_H = \sum_{n=1}^{\infty} h_n |a_n|.$$

Thus the word “hybrid” refers to the intersection of the two stated multiplier components; it is not an unspecified weighted ℓ_1 space. The diagonal map

$$J : H \rightarrow \ell_1, \quad (Ja)_n = h_n a_n,$$

is an onto linear isometry. In particular $f_n = h_n^{-1} e_n$ is the normalized coordinate sequence in H .

Whenever a sequence $(g_n) \subset H$ is used below, its probability-coefficient hull means

$$\mathcal{S}(g_n) = \left\{ \sum_{n=1}^{\infty} \alpha_n g_n : \alpha_n \geq 0, \sum_{n=1}^{\infty} \alpha_n = 1 \right\},$$

where the series is required to converge in H . In all applications the generator norms are uniformly bounded, so convergence follows from $\alpha \in \ell_1$. This is a closed infinite probability simplex only after the lower coefficient estimate is proved; it is not merely the algebraic convex hull of the generators.

Let C be a nonempty bounded convex subset of a Banach space. A sequence $(x_r) \subset C$ is an approximate fixed point sequence for $T : C \rightarrow C$ if

$$\lim_{r \rightarrow \infty} \|x_r - Tx_r\| = 0.$$

The map is nonexpansive when $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is asymptotically nonexpansive when there are $k_m \geq 1$ with $k_m \rightarrow 1$ such that $\|T^m x - T^m y\| \leq k_m \|x - y\|$. These notions originate in the classical fixed point theory represented by [1, 2]; uniform control of iterate Lipschitz constants was developed in [3]. Every nonexpansive self-map of a closed C has such a sequence. Indeed, for $0 < t < 1$ the contraction $T_t x = (1 - t)x_0 + tTx$ has a fixed point x_t , and

$$0 \leq \|x_t - Tx_t\| \leq (1 - t) \text{diam } C.$$

If T is affine, the Cesaro averages $y_N = N^{-1} \sum_{j=0}^{N-1} T^j x$ form an approximate fixed point sequence, since $Ty_N - y_N = N^{-1}(T^N x - x)$.

We shall repeatedly use the following splitting identity. If (u_r) is bounded in ℓ_1 and converges coordinatewise to $u \in \ell_1$, then

$$\limsup_{r \rightarrow \infty} \|u_r - y\|_1 = \limsup_{r \rightarrow \infty} \|u_r - u\|_1 + \|u - y\|_1. \quad (1)$$

To verify it, write $z_r = u_r - u$ and first take $w = u - y$ with finite support F . Coordinate convergence gives

$$\lim_{r \rightarrow \infty} (\|z_r + w\|_1 - \|z_r\|_1) = \sum_{j \in F} |w_j|.$$

For general w , replace it by a finite truncation w^F and use

$$\left| \|z_r + w\|_1 - \|z_r\|_1 - \|w\|_1 \right| \leq \left| \|z_r + w^F\|_1 - \|z_r\|_1 - \|w^F\|_1 \right| + 2\|w - w^F\|_1.$$

The identity follows by taking the upper limit and then enlarging F . For $Q(y) = \limsup_r \|u_r - y\|_1$, nonexpansiveness and the approximate fixed point condition also give $Q(Ty) \leq Q(y)$.

2.1. Origin and elementary structure of the hybrid weight

For completeness, we record why the sum defining h_n is the natural weight for the present intersection. Let

$$X_q = \left\{ a : \sum_{n \geq 1} \frac{n^q}{|v_n|} |a_n| < \infty \right\}, \quad X_{-q} = \left\{ a : \sum_{n \geq 1} \frac{1}{n^q |v_n|} |a_n| < \infty \right\}.$$

On $X_q \cap X_{-q}$ take the sum norm. Then

$$\|a\|_{X_q} + \|a\|_{X_{-q}} = \sum_{n \geq 1} \left(\frac{n^q}{|v_n|} + \frac{1}{n^q |v_n|} \right) |a_n| = \|a\|_H.$$

Thus no estimate is lost when the two components are represented by the single hybrid weight. If $(a^{(r)})$ is Cauchy in H , then $(Ja^{(r)})$ is Cauchy in ℓ_1 and converges to some b . Defining $a_n = b_n/h_n$ gives $a \in H$ and $a^{(r)} \rightarrow a$. Hence H is complete.

The normalized vectors have two properties used below:

$$\|f_n\|_H = 1, \quad \left\| \sum_{n \geq 1} c_n f_n \right\|_H = \sum_{n \geq 1} |c_n|.$$

Consequently, if $0 < \underline{w} \leq w_n \leq \bar{w} < \infty$, the coefficient simplex generated by $w_n f_n$ is closed. Indeed, convergence of its points in H implies convergence of their coefficient vectors in ℓ_1 , because

$$\underline{w} \sum_n |\alpha_n - \beta_n| \leq \left\| \sum_n (\alpha_n - \beta_n) w_n f_n \right\|_H \leq \bar{w} \sum_n |\alpha_n - \beta_n|.$$

Nonnegativity and unit total mass pass to the ℓ_1 limit.

2.2. Coordinate limits and loss of mass

Let $x_r = \sum_n \alpha_n^{(r)} w_n f_n$ belong to such a simplex. A diagonal selection gives a subsequence for which every $\alpha_n^{(r)}$ converges to a number $\gamma_n \geq 0$. Fatou's lemma gives $\sum_n \gamma_n \leq 1$. We write

$$\delta = 1 - \sum_n \gamma_n.$$

The number δ measures mass that escapes to indices tending to infinity. It is not a norm defect: when $\delta > 0$, the sequence cannot converge in norm to its coordinatewise limit. This observation explains why compactness cannot simply be assumed in the nonreflexive setting.

Suppose $u = \sum \gamma_n w_n f_n$ and $y = \sum \alpha_n w_n f_n$. With $c_n = \alpha_n - \gamma_n$ we have $\sum_n c_n = \delta$. Writing $c_n = c_n^+ - c_n^-$ and $N(c) = \sum_n c_n^-$ gives

$$\sum_n c_n^+ = \delta + N(c), \quad \sum_n |c_n| = \delta + 2N(c).$$

These identities are the basic bookkeeping behind all finite-perturbation arguments of Goebel–Kuczumow and Kaczor–Prus type [4, 5]. They show, for example, that any negative coefficient creates twice its mass as an excess over the smallest possible ℓ_1 distance.

2.3. The radius inequality

We include the complete upper-limit calculation because it is frequently the decisive step. Let (x_r) be an AFPS for a nonexpansive map. Then

$$\begin{aligned} Q(Ty) &= \limsup_{r \rightarrow \infty} \|x_r - Ty\| \\ &\leq \limsup_{r \rightarrow \infty} (\|x_r - Tx_r\| + \|Tx_r - Ty\|) \\ &\leq \lim_{r \rightarrow \infty} \|x_r - Tx_r\| + \limsup_{r \rightarrow \infty} \|x_r - y\| = Q(y). \end{aligned}$$

For an asymptotically nonexpansive map with constants $k_m \rightarrow 1$, fix m . The telescoping estimate

$$\|T^m x_r - x_r\| \leq \left(\sum_{j=0}^{m-1} k_j \right) \|Tx_r - x_r\|$$

shows that its left-hand side tends to zero as $r \rightarrow \infty$. Therefore

$$\limsup_{r \rightarrow \infty} \|x_r - T^m y\| \leq k_m \limsup_{r \rightarrow \infty} \|x_r - y\|.$$

Notice that m is held fixed in the first limit. Only after the radius estimate has been obtained do we let m tend to infinity.

2.4. Scope of subset theorems

A fixed point theorem for one explicitly constructed simplex is not a fixed point property of H . The latter would quantify over every nonempty closed, bounded and convex subset. Likewise, an isometry can transport a mapping only between sets that it maps onto each other. These elementary logical distinctions are important when positive results on finite perturbations are compared with fixed-point-free shifts on other subsets.

2.5. Variation-coupled geometry

For $0 \leq \theta_n \leq 1$ and $\lambda > 0$ define

$$\|a\|_{\lambda, \theta} = \|Ja\|_1 + \lambda \sum_{n \geq 1} |(Ja)_n - \theta_n(Ja)_{n+1}|.$$

This norm is equivalent to $\|\cdot\|_H$, with $\|a\|_H \leq \|a\|_{\lambda, \theta} \leq (1 + 2\lambda)\|a\|_H$. Let

$$\Delta = \{b \in \ell_1 : b_n \geq 0 \ (n \geq 1), \sum_{n=1}^{\infty} b_n = 1\}$$

be the probability simplex, let $R(b_1, b_2, \dots) = (0, b_1, b_2, \dots)$, and define

$$\Sigma_H = J^{-1}\Delta = \left\{a \in H : h_n a_n \geq 0 \ (n \geq 1), \sum_{n=1}^{\infty} h_n a_n = 1\right\}.$$

The coordinate inequalities and the continuous mass functional show that Δ is closed and convex in ℓ_1 ; it is bounded because every element has norm one. Hence Σ_H is a nonempty closed bounded convex subset of H . On this explicitly defined set put $\mathcal{R} = J^{-1}RJ$. If $Z = \{d \in \ell_1 : \sum d_n = 0\}$, its exact m th iterate coefficient is

$$L_m = \sup_{d \in Z \setminus \{0\}} \frac{\|d\|_1 + \lambda[\theta_m|d_1| + \sum_{k \geq 1} |d_k - \theta_{m+k}d_{k+1}|]}{\|d\|_1 + \lambda \sum_{k \geq 1} |d_k - \theta_k d_{k+1}|}. \quad (2)$$

Every $d \neq 0$ in Z is a positive multiple of a difference of two probability vectors, so the displayed supremum is exactly the Lipschitz coefficient, not only an upper estimate.

3. Results

3.1. Periodic profiles

Assume $\theta_{n+p} = \theta_n$ for a period p .

Theorem 3.1. *The shift $\mathcal{R}$ is asymptotically nonexpansive if and only if $\theta_1 = \dots = \theta_p = 0$.*

Proof. If the profile is zero, (2) gives $L_m = 1$. Conversely, suppose $\theta_r > 0$ for some residue r . Taking infinitely many $m \equiv r \pmod{p}$ keeps the boundary coefficient θ_m equal to θ_r . A zero-sum test vector with first coordinate nonzero and its compensating coordinate chosen beyond one full period yields

$$L_m \geq 1 + \frac{\lambda \theta_r}{2 + 2\lambda} > 1$$

on that subsequence. Hence L_m cannot converge to one. $\square$

More accurate certified lower bounds are obtained by maximizing (2) over zero-sum vectors supported in two consecutive periods. This is a finite-dimensional polyhedral problem and depends only on the period $(\theta_1, \dots, \theta_p)$.

3.2. Eventually periodic couplings

Theorem 3.2. *If θ is eventually periodic, the shift is asymptotically nonexpansive if and only if $\theta_n = 0$ for every $n \geq 1$.*

Proof. If the periodic tail contains a positive entry, translate a zero-sum test vector through multiples of the period. The same positive graph defect recurs, so L_m cannot tend to one. If the tail is zero, only finitely many original graph edges remain. Suppose that one of them is nonzero, and let r be the largest index for which $\theta_r > 0$. For all $m > r$, the numerator in (2) is $(1 + \lambda)\|d\|_1$. Choose $d_1 = \dots = d_{r+1} = 1$ and place the compensating entry $-(r + 1)$ at an index beyond $r + 1$. The variation term in the denominator is then

$$\sum_{k=1}^r (1 - \theta_k) + 1 + (r + 1) = 2r + 2 - \sum_{k=1}^r \theta_k < 2r + 2 = \|d\|_1.$$

Consequently the denominator in (2) is strictly smaller than $(1 + \lambda)\|d\|_1$, and hence $L_m > 1$ for every $m > r$. Thus asymptotic nonexpansiveness forces the finite initial part to vanish as well. The converse follows immediately from (2). $\square$

Corollary 3.3. *An eventually zero coupling is asymptotically nonexpansive if and only if it is identically zero.*

3.3. A computable periodic lower bound

For a period p , restrict (2) to vectors supported in $\{1, \dots, 2p + 1\}$. The resulting maximum is a finite linear program and is a certified lower bound for every iterate in the corresponding residue class. A positive value for one residue proves failure of asymptotic nonexpansiveness without examining arbitrary infinite-support vectors.

3.3.1. Residue-class coefficients

For a period p , the sequence (L_m) has at most p distinct values once the boundary term is grouped by m modulo p . Consequently, convergence of (L_m) is equivalent to equality of all residue-class values with one. The zero-profile theorem shows that this can occur only when every entry of the period vanishes. This finite reduction makes periodic profiles useful as exact test cases for implementations of the variational formula.

3.3.2. Constant profiles as period one

When $p = 1$, the periodic theorem reduces to the constant-coupling case. The exact coefficient then equals

$$1 + \frac{\lambda t}{2 + \lambda(2 - t)}.$$

Thus the residue-class obstruction is not merely qualitative: in the shortest period it agrees with the sharp variational value. For longer periods the finite linear programs associated with the residue classes play the same role. They distinguish which position in the period generates the largest boundary defect and provide a sharper conclusion than the universal lower bound used in the classification proof.

4. Discussion

The diagonal isometry is used to calculate distances on the explicitly defined simplex Σ_H ; it does not identify all convex subsets of H . The classification therefore concerns the barycentric shift on this simplex with the variation-coupled norm. The obstruction has two distinct forms. A nonzero periodic tail repeatedly contributes a boundary term along one residue class, while a nonzero finite initial part permanently reduces the denominator of the variational quotient for a suitable zero-sum vector. Thus deleting the tail does not by itself restore asymptotic nonexpansiveness.

The estimates are stable under multiplication of the hybrid weight by a positive scalar, since normalization by J removes that scalar. They are also unaffected by the particular representation of h_n once positivity is fixed. In contrast, the adjacent parameters remain visible in (2) and change the optimal iterate constants.

From a computational viewpoint, the formulas reduce infinite-dimensional questions to zero-sum coefficient vectors. Finite supports provide rigorous lower bounds, while the norm inequalities give global upper bounds. This makes the special cases suitable for numerical verification without replacing the analytic proof.

5. Conclusion

For the barycentric shift on Σ_H , a periodic or eventually periodic adjacent-coupling profile produces asymptotic nonexpansiveness only in the uncoupled case. Formula (2) isolates the persistent defect and reduces periodic lower bounds to finitely many residue classes. Restricting the test vectors to two consecutive periods gives certified finite-dimensional approximations, while the period-one case yields the exact coefficient. Further work may seek sharp residue-class coefficients for longer periods and extend the variational approach to nonperiodic coupling profiles.

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Conflict of interest

The author declares no conflict of interest.

Data availability

No data sets were generated or analyzed in this theoretical study.

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