TURKISH JOURNAL OF SCIENCE
VOLUME 11, ISSUE 2, 83–89
ISSN: 2587–0971
https://engineering-journals.org

Affine Asymptotically Nonexpansive Mappings on Finite-Gap Subsets of a Hybrid Sequence Space

Veysel Nezir, Nizami Mustafa

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Türkiye

Abstract. We investigate fixed points of affine mappings on a class of explicitly constructed probability simplices in a nonreflexive hybrid sequence space. The simplices are generated by a bounded coefficient profile whose minimum is attained at finitely many indices and is separated from all remaining values by a positive gap. A diagonal isometry transfers the problem to the space of absolutely summable sequences, where coordinatewise limits of approximate fixed point sequences can lose mass. We derive a coercive distance estimate that confines any lost mass to the finite set of minimum-weight coordinates. This estimate shows that a suitable orbit approaches a compact minimizing simplex and is therefore relatively compact. The closed convex hull of that orbit is compact and invariant, so a fixed point follows from Schauder's theorem. We also obtain a quantitative stability result that controls the distance to the minimizing simplex through the excess asymptotic radius. The proof does not assume that the ambient space has the fixed point property and keeps the conclusion restricted to the displayed finite-gap subsets.

1. Introduction

The fixed point theory of nonexpansive mappings began with geometric existence theorems such as Kirk's theorem [1]. Goebel and Kirk introduced asymptotically nonexpansive mappings [2] and subsequently studied mappings whose iterates have a common Lipschitz bound [3]. For nonreflexive spaces these global properties can fail, so one is led to ask which explicitly described closed bounded convex subsets retain a fixed point theorem. Goebel and Kuczumow constructed irregular simplices with this property [4]; Kaczor and Prus later studied related finite-perturbation and finite-minimum configurations in ℓ_1 [5]. Alspach's example [6] shows at the same time that a weakly compact convex subset of $L^1[0,1]$ may support a fixed-point-free nonexpansive map.

Cesaro and generalized difference matrix domains provide a separate source of weighted coordinate spaces. Their Kothe–Toeplitz multiplier duals were computed in several settings; see [7, 8]. The hybrid space used here is the intersection of two explicitly stated multiplier components. The definition is given

Corresponding author: Veysel Nezir, Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Türkiye, mail address: veyselnezir@yahoo.com (ORCID: 0000-0001-9640-8526)
Received: 05 October 2026; Accepted: 12 October 2026; Published: 13 October 2026
Keywords. Köthe–Toeplitz duality; probability simplex; coefficient geometry; compactness argument; nonreflexive Banach space; asymptotic regularity
2010 *Mathematics Subject Classification*. 47H10, 46B45, 46A45
Cited this article as: Nezir, V., Mustafa, N., Affine asymptotically nonexpansive mappings on finite-gap subsets of a hybrid sequence space, Turkish Journal of Science, 11(2), 2026, 83–89
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below from the boundary-anchored Cesaro–difference domain, rather than introduced as an unexplained weighted ℓ_1 space. The diagonal normalization $J : H \to \ell_1$ is a proof device: all subsets, mappings and norm estimates are defined first and transported only through an explicitly displayed isometry.

We study bounded coefficient profiles with finitely many minimum indices separated by a positive gap and derive an affine asymptotically nonexpansive fixed point theorem from a quantitative orbit estimate.

The fixed point property of a Banach space quantifies over every nonempty closed bounded convex subset. None of the subset theorems below makes that global assertion. Each conclusion is tied to its displayed probability simplex and to the stated mapping class. This distinction is essential when the results are compared with fixed-point-free shifts or function-space counterexamples.

2. Methods and mathematical framework

Let ω be the space of all scalar sequences, let $q \geq 1$, and let $v = (v_n)$ have no zero terms. With the forward-difference convention, set

$$(\Delta_v^q x)_n = \Delta^q (vx)_n = \sum_{j=0}^{q} (-1)^j \binom{q}{j} v_{n+j} x_{n+j}.$$

Let C denote the Cesaro matrix, $(Cz)_n = n^{-1} \sum_{k=1}^{n} z_k$, and define the boundary-anchored domain

$$\mathcal{E}_{C,v}^q = \{x \in \omega : C\Delta_v^q x \in \ell_\infty\},$$

equipped with

$$\|x\|_{\mathcal{E}} = \max\left\{ \max_{1 \leq j \leq q} |v_j x_j|, \|C\Delta_v^q x\|_\infty \right\}.$$

Generalized difference domains and their duals are studied in [7, 8]. The boundary block is necessary: the transform vanishes whenever $v_n x_n$ is a polynomial of degree at most $q-1$. The Kothe–Toeplitz alpha-dual of a sequence space X is

$$X^\alpha = \left\{ a : \sum_{n \geq 1} |a_n x_n| < \infty \text{ for every } x \in X \right\}.$$

For the boundary-anchored domain one obtains the multiplier dual

$$U_1^q(v) = (\mathcal{E}_{C,v}^q)^\alpha = \ell_1(u), \qquad u_n = \frac{n^q}{|v_n|}.$$

The complementary Cesaro–difference inversion component is

$$W_1^q(v) = \ell_1(w), \qquad w_n = \frac{1}{n^q |v_n|}.$$

We define the hybrid dual and its weight by

$$H = H_1^{q,\#}(v) = U_1^q(v) \cap W_1^q(v) = \ell_1(h), \qquad h_n = u_n + w_n = \frac{n^{2q}+1}{n^q |v_n|},$$

with the sum norm

$$\|a\|_H = \sum_{n=1}^{\infty} h_n |a_n|.$$

Thus the word "hybrid" refers to the intersection of the two stated multiplier components; it is not an unspecified weighted ℓ_1 space. The diagonal map

$$J : H \to \ell_1, \qquad (Ja)_n = h_n a_n,$$

is an onto linear isometry. In particular $f_n = h_n^{-1} e_n$ is the normalized coordinate sequence in H.

Whenever a sequence $(g_n) \subset H$ is used below, its probability-coefficient hull means

$$\mathcal{S}(g_n) = \left\{ \sum_{n=1}^{\infty} \alpha_n g_n : \alpha_n \geq 0,\ \sum_{n=1}^{\infty} \alpha_n = 1 \right\},$$

where the series is required to converge in H. In all applications the generator norms are uniformly bounded, so convergence follows from $\alpha \in \ell_1$. This is a closed infinite probability simplex only after the lower coefficient estimate is proved; it is not merely the algebraic convex hull of the generators.

Let C be a nonempty bounded convex subset of a Banach space. A sequence $(x_r) \subset C$ is an approximate fixed point sequence for $T : C \to C$ if

$$\lim_{r\to\infty} \|x_r - Tx_r\| = 0.$$

The map is nonexpansive when $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is asymptotically nonexpansive when there are $k_m \geq 1$ with $k_m \to 1$ such that $\|T^m x - T^m y\| \leq k_m \|x - y\|$. These notions originate in the classical fixed point theory represented by [1, 2]; uniform control of iterate Lipschitz constants was developed in [3]. Every nonexpansive self-map of a closed C has such a sequence. Indeed, for $0 < t < 1$ the contraction $T_t x = (1 - t)x_0 + tTx$ has a fixed point x_t, and

$$0 \leq \|x_t - Tx_t\| \leq (1 - t) \operatorname{diam} C.$$

If T is affine, the Cesaro averages $y_N = N^{-1} \sum_{j=0}^{N-1} T^j x$ form an approximate fixed point sequence, since $Ty_N - y_N = N^{-1}(T^N x - x)$.

We shall repeatedly use the following splitting identity. If (u_r) is bounded in ℓ_1 and converges coordinatewise to $u \in \ell_1$, then

$$\limsup_{r\to\infty} \|u_r - y\|_1 = \limsup_{r\to\infty} \|u_r - u\|_1 + \|u - y\|_1. \tag{1}$$

To verify it, write $z_r = u_r - u$ and first take $w = u - y$ with finite support F. Coordinate convergence gives

$$\lim_{r\to\infty} (\|z_r + w\|_1 - \|z_r\|_1) = \sum_{j\in F} |w_j|.$$

For general w, replace it by a finite truncation w^F and use

$$\left| [\|z_r + w\|_1 - \|z_r\|_1] - \|w\|_1 \right| \leq \left| [\|z_r + w^F\|_1 - \|z_r\|_1] - \|w^F\|_1 \right| + 2\|w - w^F\|_1.$$

The identity follows by taking the upper limit and then enlarging F. For $Q(y) = \limsup_r \|u_r - y\|_1$, nonexpansiveness and the approximate fixed point condition also give $Q(Ty) \leq Q(y)$.

2.1. Origin and elementary structure of the hybrid weight

For completeness, we record why the sum defining h_n is the natural weight for the present intersection. Let

$$X_q = \left\{ a : \sum_{n\geq 1} \frac{n^q}{|v_n|} |a_n| < \infty \right\}, \qquad X_{-q} = \left\{ a : \sum_{n\geq 1} \frac{1}{n^q |v_n|} |a_n| < \infty \right\}.$$

On $X_q \cap X_{-q}$ take the sum norm. Then

$$\|a\|_{X_q} + \|a\|_{X_{-q}} = \sum_{n\geq 1} \left(\frac{n^q}{|v_n|} + \frac{1}{n^q |v_n|} \right) |a_n| = \|a\|_H.$$

Thus no estimate is lost when the two components are represented by the single hybrid weight. If $(a^{(r)})$ is Cauchy in H, then $(Ja^{(r)})$ is Cauchy in ℓ_1 and converges to some b. Defining $a_n = b_n / h_n$ gives $a \in H$ and $a^{(r)} \to a$. Hence H is complete.

The normalized vectors have two properties used below:

$$\|f_n\|_H = 1, \qquad \left\|\sum_{n\geq 1} c_n f_n\right\|_H = \sum_{n\geq 1} |c_n|.$$

Consequently, if $0 < \underline{w} \leq w_n \leq \overline{w} < \infty$, the coefficient simplex generated by $w_n f_n$ is closed. Indeed, convergence of its points in H implies convergence of their coefficient vectors in ℓ_1, because

$$\underline{w} \sum_n |\alpha_n - \beta_n| \leq \left\|\sum_n (\alpha_n - \beta_n) w_n f_n\right\|_H \leq \overline{w} \sum_n |\alpha_n - \beta_n|.$$

Nonnegativity and unit total mass pass to the ℓ_1 limit.

2.2. Coordinate limits and loss of mass

Let $x_r = \sum_n \alpha_n^{(r)} w_n f_n$ belong to such a simplex. A diagonal selection gives a subsequence for which every $\alpha_n^{(r)}$ converges to a number $\gamma_n \geq 0$. Fatou's lemma gives $\sum_n \gamma_n \leq 1$. We write

$$\delta = 1 - \sum_n \gamma_n.$$

The number δ measures mass that escapes to indices tending to infinity. It is not a norm defect: when $\delta > 0$, the sequence cannot converge in norm to its coordinatewise limit. This observation explains why compactness cannot simply be assumed in the nonreflexive setting.

Suppose $u = \sum \gamma_n w_n f_n$ and $y = \sum \alpha_n w_n f_n$. With $c_n = \alpha_n - \gamma_n$ we have $\sum_n c_n = \delta$. Writing $c_n = c_n^+ - c_n^-$ and $N(c) = \sum_n c_n^-$ gives

$$\sum_n c_n^+ = \delta + N(c), \qquad \sum_n |c_n| = \delta + 2N(c).$$

These identities are the basic bookkeeping behind all finite-perturbation arguments of Goebel–Kuczumow and Kaczor–Prus type [4, 5]. They show, for example, that any negative coefficient creates twice its mass as an excess over the smallest possible ℓ_1 distance.

2.3. The radius inequality

We include the complete upper-limit calculation because it is frequently the decisive step. Let (x_r) be an AFPS for a nonexpansive map. Then

$$\begin{aligned} Q(Ty) &= \limsup_{r\to\infty} \|x_r - Ty\| \\ &\leq \limsup_{r\to\infty} (\|x_r - Tx_r\| + \|Tx_r - Ty\|) \\ &\leq \lim_{r\to\infty} \|x_r - Tx_r\| + \limsup_{r\to\infty} \|x_r - y\| = Q(y). \end{aligned}$$

For an asymptotically nonexpansive map with constants $k_m \to 1$, fix m. The telescoping estimate

$$\|T^m x_r - x_r\| \leq \left(\sum_{j=0}^{m-1} k_j\right) \|Tx_r - x_r\|$$

shows that its left-hand side tends to zero as $r \to \infty$. Therefore

$$\limsup_{r\to\infty} \|x_r - T^m y\| \leq k_m \limsup_{r\to\infty} \|x_r - y\|.$$

Notice that m is held fixed in the first limit. Only after the radius estimate has been obtained do we let m tend to infinity.

2.4. Scope of subset theorems

A fixed point theorem for one explicitly constructed simplex is not a fixed point property of H. The latter would quantify over every nonempty closed, bounded and convex subset. Likewise, an isometry can transport a mapping only between sets that it maps onto each other. These elementary logical distinctions are important when positive results on finite perturbations are compared with fixed-point-free shifts on other subsets.

3. Results

3.1. The finite-gap class

Let $a_n \geq 0$ be bounded, $a_* = \inf a_n$, and assume $N_0 = \{n : a_n = a_*\}$ is nonempty and finite while $\rho = \inf_{n\notin N_0} a_n - a_* > 0$. Put $G_n = (1 + a_n)f_n$ and $C_H(a) = \{\sum t_n G_n : t_n \geq 0, \sum t_n = 1\}$.

Theorem 3.1. *Every affine asymptotically nonexpansive self-map of $C_H(a)$ has a fixed point.*

Proof. Conjugate to $F_n = (1 + a_n)e_n$ in ℓ_1. Let (x_r) be the Cesaro AFPS, $x_r \to u = \sum \gamma_n F_n$ coordinatewise, and $\sum \gamma_n = 1 - \delta$. For fixed m, telescoping gives $\lim_r \|T^m x_r - x_r\|_1 = 0$, hence $Q(T^m y) \leq k_m Q(y)$, where $k_m \to 1$.

If $\delta = 0$, splitting yields $0 \leq \|T^m u - u\|_1 \leq (k_m - 1)Q(u)$; the triangle inequality then gives $Tu = u$. Suppose $\delta > 0$. With $w_n = 1 + a_n$, $w_* = 1 + a_*$, and $c_n = \alpha_n - \gamma_n$, direct summation gives

$$\|y - u\|_1 - w_*\delta \geq 2w_* \sum c_n^- + \rho \sum_{n\notin N_0} |c_n|.$$

The minimizing set $\Lambda = u + \{\sum_{j\in N_0} s_j F_j : s_j \geq 0, \sum s_j = \delta\}$ is compact. For $z \in \Lambda$, the radius inequality shows $\|T^m z - u\|_1 - w_*\delta \to 0$; the displayed coercive estimate gives $\operatorname{dist}(T^m z, \Lambda) \to 0$. Thus the orbit is relatively compact. Its closed convex hull is compact, convex and invariant under the affine continuous map T. Schauder's theorem completes the proof. □

3.2. Quantitative stability of the finite-gap minimum

Theorem 3.2. *Let $u = \sum \gamma_n G_n$ have coefficient mass $1 - \delta$, where $\delta > 0$, and let Λ_u be the finite minimizing simplex. If $y_r \in C_H(a)$ and $\|y_r - u\|_H \to (1 + a_*)\delta$, then $\operatorname{dist}(y_r, \Lambda_u) \to 0$.*

Proof. Use normalized coordinates and put $w_n = 1 + a_n$, $w_* = 1 + a_*$. For the coefficient differences $c^{(r)}$ the coercive identity yields

$$0 \leq 2w_* N(c^{(r)}) + \rho \sum_{n\notin N_0} |c_n^{(r)}| \leq \|y_r - u\|_1 - w_*\delta.$$

Both terms tend to zero. Let $P_r = \sum_{j\in N_0} (c_j^{(r)})^+$. Then $P_r \to \delta$. For large r set

$$s_j^{(r)} = \delta (c_j^{(r)})^+ / P_r \quad (j \in N_0), \qquad z_r = u + \sum_{j\in N_0} s_j^{(r)} G_j.$$

Thus $z_r \in \Lambda_u$. If $W = \sup_n w_n$, direct summation gives

$$\|y_r - z_r\|_H \leq 2W \left[N(c^{(r)}) + \sum_{n\notin N_0} |c_n^{(r)}| \right] \longrightarrow 0.$$

□

Corollary 3.3. *Under the hypotheses of the main theorem, the orbit used in its proof is totally bounded. Hence its closed convex hull is norm compact.*

The explicit estimate is useful when the fixed point argument is perturbed: it controls how much coefficient mass may lie outside the finite minimizing set in terms of the excess radius.

3.2.1. A uniform orbit estimate

For $z \in \Lambda_u$, the proof gives more than relative compactness:

$$\|T^m z - u\|_1 - w_* \delta \leq (k_m - 1)(Q(u) + w_* \delta).$$

Combining this inequality with the coercive estimate bounds the negative coefficient mass and the mass outside N_0 by an explicit multiple of $k_m - 1$. Hence a known rate for $k_m \downarrow 1$ produces a rate at which the orbit approaches the compact minimizing simplex. This quantitative form is useful even though Schauder's theorem itself gives no rate for the location of the final fixed point.

3.2.2. The zero-defect alternative

When the coordinatewise AFPS limit loses no mass, the finite-gap condition is not needed: the splitting identity alone gives convergence of $T^m u$ to u. The gap enters only in the positive-defect case, where it confines the missing mass to finitely many minimum-weight coordinates. This separation is useful in applications because one may first test whether the AFPS is tight in coefficient space. If it is tight, the direct argument applies; if it is not, the finite-gap coercivity supplies the compact replacement. The theorem therefore combines two logically distinct routes to a fixed point rather than applying compactness indiscriminately.

4. Discussion

The diagonal isometry is used here to calculate distances, not to identify all convex subsets of H with one another. The conclusion therefore remains attached to the set defined in the statement. This point also explains why fixed-point-free affine shifts on an unperturbed probability simplex do not contradict the fixed point theorems for finite perturbations. In the latter case, loss of mass is forced back onto finitely many coordinates of minimal weight; in the former case, the missing mass can continue to move to infinity.

The estimates are stable under multiplication of the hybrid weight by a positive scalar, since normalization by J removes that scalar. They are also unaffected by the particular representation of h_n once positivity is fixed. What matters for the diagonal results is the coefficient geometry; for the variation-coupled results, the adjacent parameters retain additional information and therefore change the optimal iterate constants.

From a computational viewpoint, the formulas reduce infinite-dimensional questions to zero-sum coefficient vectors. Finite supports provide rigorous lower bounds, while the norm inequalities give global upper bounds. This makes the special cases suitable for numerical verification without replacing the analytic proof.

5. Conclusion

The argument gives a self-contained result for the stated class in the hybrid space H. It also indicates which assumptions are structural: finite minimum sets create compact minimizing simplices, whereas unrestricted infinite blocks permit fixed-point-free shifts. Further work may investigate quantitative perturbations and computational bounds for nonstationary weights.

Acknowledgements

Not applicable.

Conflict of interest

The authors declare no conflict of interest.

Ethical approval

Not applicable. This research did not involve human participants or animals.

Data availability

No data were generated or analysed in this theoretical study.

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