

A Caputo-Fabrizio Fractional Integral Approach to Hermite-Hadamard Type Inequalities for Strongly (s, m) -Convex Functions

Ebru Karaduman^a, Kader Aşak^b

^aDepartment of Mathematics, Faculty of Science and Letters, Ağrı İbrahim Çeçen University, 04100, Ağrı, Turkey

^bInstitute of Graduate Studies, Department of Mathematics, Ağrı İbrahim Çeçen University, Ağrı, Turkey

Abstract. In this paper, we consider the integral operators corresponding to the non-singular fractional derivative introduced by Caputo and Fabrizio in 2015. Using the Caputo-Fabrizio fractional integrals, new Hermite-Hadamard type inequalities are established for strongly (s, m) -convex functions. The obtained results not only generalize several classical inequalities available in the literature but also highlight the advantages of the non-singular kernel structure in the study of strongly convex mappings. Furthermore, the theoretical findings are supported by various applications. In addition, the graphs prepared to visualize the results clearly demonstrate the potential of the proposed approach in both theoretical and applied mathematics.

1. Introduction

The modern development of convexity theory is often traced back to a result established by Charles Hermite in 1881, which was published without proof in 1883 in the journal *Mathesis*. Subsequently, Jacques Hadamard extended these ideas in 1893 through his celebrated inequality for convex functions. Nevertheless, the theory of convex functions had not yet attained a systematic form. The rigorous and systematic investigation of convexity began with the seminal works of J. L. W. V. Jensen during 1905-1906, which laid the foundations of convex analysis as an independent mathematical discipline [10, 11].

Since the definition of a convex function is expressed through an inequality, inequalities occupy a central position in convex analysis. Consequently, many fundamental results in the theory are formulated in terms of inequalities. Among these, Jensen's inequality stands out as one of the most influential results and has found extensive applications in probability theory, optimization, economics, functional analysis, and statistics.

The theory of inequalities is a branch of mathematics concerned with the formulation, analysis, and application of inequalities. It constitutes one of the fundamental components of mathematical analysis and plays an essential role in both pure and applied mathematics. The foundations of inequality theory were established during the eighteenth and nineteenth centuries through the contributions of numerous

Corresponding author: Ebru Karaduman, Department of Mathematics, Faculty of Science and Letters, Ağrı İbrahim Çeçen University, 04100, Ağrı, Turkey, mail address: eyuksel@agri.edu.tr

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distinguished mathematicians. Among the most influential contributors were P. L. Chebyshev (1821-1894), C. F. Gauss (1777-1855), and A. L. Cauchy (1789-1857). A major milestone in the development of the subject was the publication of the book *Inequalities* by G. H. Hardy, J. E. Littlewood, and G. Polya in 1934 [12]. This classical work provided the first systematic treatment of inequality theory and established a rigorous framework for the classification, analysis, and application of inequalities in a wide variety of mathematical settings. As a result, inequality theory evolved into one of the central areas of modern mathematical analysis.

Fractional calculus is a natural extension of classical differential and integral calculus, concerned with the definition and study of derivatives and integrals of arbitrary (non-integer) order. The origins of this field can be traced back to a famous correspondence between Marquis de L'Hôpital and Gottfried Wilhelm Leibniz in 1695. In this exchange, L'Hôpital asked Leibniz what the meaning of a derivative of order $1/2$ would be. Leibniz replied that, although such an operation appeared paradoxical at first sight, it might eventually lead to important consequences. This observation is widely regarded as the starting point of the theory that later developed into fractional calculus [13].

Following these early developments, Leonhard Euler investigated derivatives of arbitrary order in 1738. Subsequently, several mathematicians contributed to the gradual development of fractional calculus. In particular, Joseph Fourier, in 1822, considered representations involving derivatives of arbitrary order through integral formulations, thereby contributing to the early mathematical foundations of the subject.

A major milestone in the development of fractional calculus was the work of Niels Henrik Abel in 1826 on the tautochrone problem. In the course of solving this problem, Abel derived and solved an integral equation whose structure is now recognized as closely related to fractional integration. This work is widely regarded as one of the first significant applications of fractional calculus and played an important role in the subsequent development of the field [14, 15].

In 1832, Joseph Liouville developed a formula for the derivatives of exponential functions and introduced the formulation now known as the Liouville fractional integral. This contribution substantially strengthened the theoretical foundations of fractional calculus and paved the way for later developments by Riemann and other researchers.

Today, fractional calculus has found extensive applications in both pure and applied mathematics, as well as in diverse fields such as physics, statistics, signal processing, control theory, mathematical modeling, biology, and engineering sciences. In parallel with these developments, numerous fractional integral and derivative operators, including the Riemann–Liouville, Hadamard, Caputo, Caputo–Fabrizio, Atangana–Baleanu, and Katugampola operators, have been introduced, leading to the development of rich theoretical frameworks and a wide range of applications [16]–[21].

The various fractional integral and derivative operators developed as generalizations of the classical Riemann–Liouville and Caputo operators have enabled the reinterpretation of many fundamental inequalities in mathematical analysis and have established a strong connection between inequality theory and fractional calculus. Consequently, the interaction between these two areas has become an active and rapidly growing field of contemporary research. For recent results related to new fractional integral operators and integral inequalities, we refer the papers [1]–[9].

Definition 1.1 ([17], [18]). The left-sided and the right-sided Caputo-Fabrizio fractional integrals of order $\zeta \in (0, 1)$ of a function P are defined respectively by

$$\begin{aligned} {}_w^{CF}\mathfrak{I}^\zeta P(\kappa) &= \frac{(1-\zeta)}{M(\zeta)}P(\kappa) + \frac{\zeta}{M(\zeta)} \int_w^\kappa P(t) dt, \\ {}_z^{CF}\mathfrak{I}^\zeta P(\kappa) &= \frac{(1-\zeta)}{M(\zeta)}P(\kappa) + \frac{\zeta}{M(\zeta)} \int_\kappa^z P(t) dt, \end{aligned}$$

where $M(\zeta)$ is a normalization function satisfying $M(0) = M(1) = 1$.

Definition 1.2 ([11]). The function $P : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if we have

$$P(tx + (1-t)y) \leq tP(x) + (1-t)P(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.3 ([23]). The function $P : [0, \infty) \rightarrow \mathbb{R}$ is said to be an s -convex function in the second sense if

$$P(tx + (1-t)y) \leq t^s P(x) + (1-t)^s P(y)$$

for all $x, y \geq 0$, $t \in [0, 1]$ and $s \in (0, 1]$.

Definition 1.4 ([24]). The function $P : [0, b] \rightarrow \mathbb{R}$ is said to be an m -convex function, where $m \in (0, 1]$, if

$$P(tx + m(1-t)y) \leq tP(x) + m(1-t)P(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Definition 1.5 ([25]). The function $P : [0, b] \rightarrow \mathbb{R}$ is said to be an (s, m) -convex function, where $m \in (0, 1]$ and $s \in (0, 1]$, if

$$P(tx + m(1-t)y) \leq t^s P(x) + m(1-t)^s P(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Definition 1.6 ([27]). Let I be a nonempty convex subset of a normed space. A real-valued function P is said to be strongly convex with modulus $Z > 0$ if

$$P(tx + (1-t)y) \leq tP(x) + (1-t)P(y) - Zt(1-t)\|x - y\|^2$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.7 ([27]). The function $P : [0, \infty) \rightarrow \mathbb{R}$ is said to be a strongly s -convex function with modulus $Z > 0$ if

$$P(tx + (1-t)y) \leq t^s P(x) + (1-t)^s P(y) - Zt(1-t)\|x - y\|^2$$

for all $x, y \geq 0$, $t \in [0, 1]$ and $s \in (0, 1]$.

Definition 1.8 ([28]). The function $P : [0, \infty) \rightarrow \mathbb{R}$ is said to be a strongly m -convex function with modulus $Z > 0$ if

$$P(tx + m(1-t)y) \leq tP(x) + m(1-t)P(y) - Zt(1-t)\|x - y\|^2$$

for all $x, y \geq 0$, $t \in [0, 1]$ and $m \in (0, 1]$.

Definition 1.9 ([29]). The function $P : [0, \infty) \rightarrow \mathbb{R}$ is said to be a strongly (s, m) -convex function with modulus $Z > 0$ if

$$P(tx + m(1-t)y) \leq t^s P(x) + m(1-t)^s P(y) - Zt(1-t)\|x - y\|^2$$

for all $x, y \geq 0$, $t \in [0, 1]$, $s \in (0, 1]$ and $m \in (0, 1]$.

The classical Hermite–Hadamard inequality for a convex function P on an interval $[w, z]$ states that

$$P\left(\frac{w+z}{2}\right) \leq \frac{1}{z-w} \int_w^z P(\kappa) d\kappa \leq \frac{P(w) + P(z)}{2}.$$

The following section is dedicated to deriving a novel identity for Hermite-Hadamard type inequalities that involve Caputo-Fabrizio fractional integrals for strongly (s, m) -convex functions. Subsequently, by utilizing this newly established identity along with classical integral inequalities, including the Hölder, the power-mean, the Young and the Hölder-İşcan inequalities, we derive and present several new inequalities.

2. Bounds for Hermite-Hadamard type inequalities

The lemma presented below plays a vital role in proving our remaining primary results.

Lemma 2.1. Suppose that $P : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function on I° , $w, z \in I^\circ$, $w < z$ and $P' \in L^1[w, z]$. Then, the following identity holds:

$$\begin{aligned} & {}^{CF}_w \mathfrak{J}^\zeta P(\kappa) + {}^{CF}_z \mathfrak{J}^\zeta P(\kappa) + \frac{\zeta}{M(\zeta)} \left[\frac{wP(z) - zP(w)}{2} \right] \\ & - \left[\frac{\zeta(z-w)}{2M(\zeta)} P\left(\frac{w+z}{2}\right) + \frac{2(1-\zeta)P(\kappa)}{M(\zeta)} \right] \\ & = \frac{\zeta(z-w)}{4M(\zeta)} \left[\int_0^1 (tz + (1-t)w) P' \left(\frac{1-t}{2}z + \frac{1+t}{2}w \right) dt \right. \\ & \quad \left. + \int_0^1 (tw + (1-t)z) P' \left(\frac{1-t}{2}w + \frac{1+t}{2}z \right) dt \right], \end{aligned}$$

where $\kappa \in [w, z]$, $\zeta \in (0, 1)$, $t \in [0, 1]$ and M is a normalization function.

Proof. By applying integration by parts to the integrals on the right-hand side and rearranging terms, one obtains the stated identity. The details are standard and are omitted for brevity. $\square$

Theorem 2.2. Suppose that $P : [w, z] \subset [0, \infty) \rightarrow \mathbb{R}$ is a differentiable function on (w, z) , $P' \in L^1[w, z]$, and $|P'|$ is strongly (s, m) -convex with modulus $Z > 0$. Then

$$\begin{aligned} & \left| {}^{CF}_w \mathfrak{J}^\zeta P(\kappa) + {}^{CF}_z \mathfrak{J}^\zeta P(\kappa) + \frac{\zeta}{M(\zeta)} \left[\frac{wP(z) - zP(w)}{2} \right] \right. \\ & \quad \left. - \left[\frac{\zeta(z-w)}{2M(\zeta)} P\left(\frac{w+z}{2}\right) + \frac{2(1-\zeta)P(\kappa)}{M(\zeta)} \right] \right| \\ & \leq \frac{\zeta(z-w)^2}{4M(\zeta)} \left[\frac{|P'(w)| + m|P'(z/m)|}{s+1} - \frac{Z(z-w)^2}{6} \right], \end{aligned}$$

for $\zeta \in (0, 1)$ and $s, m \in (0, 1]$.

Proof. By Lemma 2.1, employing the properties of the modulus and using strongly (s, m) -convexity of $|P'|$, we obtain the desired estimate after evaluating the resulting integrals. $\square$

Corollary 2.3. Under the same assumptions of Theorem 2.2 with $m = 1$, $s = 1$ and $Z = 0$, one recovers the classical Hermite-Hadamard type bound for convex functions via Caputo-Fabrizio fractional integrals.

Theorem 2.4. Suppose that $P : [w, z] \subset [0, \infty) \rightarrow \mathbb{R}$ is a differentiable function on (w, z) , $P' \in L^1[w, z]$, and $|P'|^q$ is strongly (s, m) -convex with modulus $Z > 0$ for $q > 1$. Then

$$\begin{aligned} & \left| {}^{CF}_w \mathfrak{J}^\zeta P(\kappa) + {}^{CF}_z \mathfrak{J}^\zeta P(\kappa) + \frac{\zeta}{M(\zeta)} \left[\frac{wP(z) - zP(w)}{2} \right] \right. \\ & \quad \left. - \left[\frac{\zeta(z-w)}{2M(\zeta)} P\left(\frac{w+z}{2}\right) + \frac{2(1-\zeta)P(\kappa)}{M(\zeta)} \right] \right| \\ & \leq \frac{\zeta(z-w)^2}{4M(\zeta)} \left(\frac{1}{p+1} \right)^{1/p} \left[\frac{|P'(w)|^q + m|P'(z/m)|^q}{s+1} - \frac{Z(z-w)^2}{6} \right]^{1/q}, \end{aligned}$$

where $1/p + 1/q = 1$.

Proof. From Lemma 2.1 and Hölder's inequality, the stated bound follows after applying strong (s, m) -convexity of $|P'|^q$. $\square$

Corollary 2.5. *Under the same assumptions of Theorem 2.4 with $m = 1$, $s = 1$ and $Z = 0$, the corresponding classical bound is recovered.*

Theorem 2.6. *Suppose that $P : [w, z] \subset [0, \infty) \rightarrow \mathbb{R}$ is a differentiable function on (w, z) , $P' \in L^1[w, z]$, and $|P'|^q$ is strongly (s, m) -convex with modulus $Z > 0$ for $q \geq 1$. Then a power-mean type estimate holds:*

$$\begin{aligned} & \left| {}^{CF}_w \mathfrak{S}^\zeta P(\kappa) + {}^{CF}_z \mathfrak{S}^\zeta P(\kappa) + \frac{\zeta}{M(\zeta)} \left[\frac{wP(z) - zP(w)}{2} \right] \right. \\ & \quad \left. - \left[\frac{\zeta(z-w)}{2M(\zeta)} P\left(\frac{w+z}{2}\right) + \frac{2(1-\zeta)P(\kappa)}{M(\zeta)} \right] \right| \\ & \leq \frac{\zeta(z-w)^2}{4M(\zeta)} \left(\frac{1}{2}\right)^{1-1/q} \left[\frac{|P'(w)|^q + m|P'(z/m)|^q}{s+1} - \frac{Z(z-w)^2}{6} \right]^{1/q}. \end{aligned}$$

Proof. From Lemma 2.1 and the power-mean inequality, the result follows by strong (s, m) -convexity of $|P'|^q$. $\square$

Corollary 2.7. *Under the same assumptions of Theorem 2.6 with $m = 1$, $s = 1$ and $Z = 0$, the classical power-mean bound is recovered.*

Theorem 2.8. *Suppose that $P : [w, z] \subset [0, \infty) \rightarrow \mathbb{R}$ is a differentiable function on (w, z) , $P' \in L^1[w, z]$, and $|P'|$ is strongly (s, m) -convex with modulus $Z > 0$. Then, by Young's inequality,*

$$\begin{aligned} & \left| {}^{CF}_w \mathfrak{S}^\zeta P(\kappa) + {}^{CF}_z \mathfrak{S}^\zeta P(\kappa) + \frac{\zeta}{M(\zeta)} \left[\frac{wP(z) - zP(w)}{2} \right] \right. \\ & \quad \left. - \left[\frac{\zeta(z-w)}{2M(\zeta)} P\left(\frac{w+z}{2}\right) + \frac{2(1-\zeta)P(\kappa)}{M(\zeta)} \right] \right| \\ & \leq \frac{\zeta(z-w)^2}{4M(\zeta)} \left[\frac{|P'(w)| + m|P'(z/m)|}{2(s+1)} + \frac{1}{2} - \frac{Z(z-w)^2}{12} \right]. \end{aligned}$$

Proof. From Lemma 2.1 and Young's inequality, the bound is obtained after applying strong (s, m) -convexity. $\square$

Corollary 2.9. *Under the same assumptions of Theorem 2.8 with $m = 1$, $s = 1$ and $Z = 0$, the corresponding classical Young-type bound is recovered.*

Theorem 2.10. *Suppose that $P : [w, z] \subset [0, \infty) \rightarrow \mathbb{R}$ is a differentiable function on (w, z) , $P' \in L^1[w, z]$, and $|P'|^q$ is strongly (s, m) -convex with modulus $Z > 0$ for $q > 1$. Then a Hölder-İşcan type estimate holds:*

$$\begin{aligned} & \left| {}^{CF}_w \mathfrak{S}^\zeta P(\kappa) + {}^{CF}_z \mathfrak{S}^\zeta P(\kappa) + \frac{\zeta}{M(\zeta)} \left[\frac{wP(z) - zP(w)}{2} \right] \right. \\ & \quad \left. - \left[\frac{\zeta(z-w)}{2M(\zeta)} P\left(\frac{w+z}{2}\right) + \frac{2(1-\zeta)P(\kappa)}{M(\zeta)} \right] \right| \\ & \leq \frac{\zeta(z-w)^2}{4M(\zeta)} \cdot \frac{1}{(p+1)^{1/p}} \left[\frac{|P'(w)|^q + m|P'(z/m)|^q}{s+1} - \frac{Z(z-w)^2}{6} \right]^{1/q}, \end{aligned}$$

with $1/p + 1/q = 1$.

Proof. From Lemma 2.1 and the Hölder-İşcan inequality, the result follows by strong (s, m) -convexity of $|P'|^q$. $\square$

Corollary 2.11. *Under the same assumptions of Theorem 2.10 with $m = 1, s = 1$ and $Z = 0$, the classical Hölder-İşcan bound is recovered.*

Example 2.12. *The following Figure 1 describes the validity of the inequality of Theorem 2.2 for $P(x) = x^3 + x^2$, $w = 0, z = 2, \zeta = 0.5$ and $Z = 3$.*

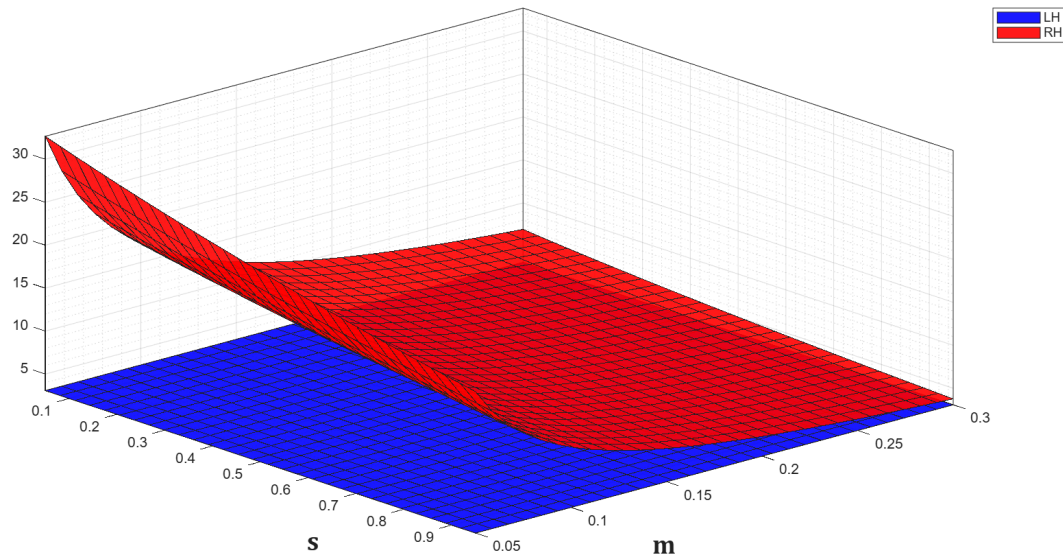


Figure 1: Graphical description for $s \in [0.05, 0.95]$ and $m \in [0.05, 0.3]$.

The following Figure 2 describes the validity of the inequality of Theorem 2.2 for $P(x) = x^3 + x^2$, $w = 0, z = 2, s = 0.25, m = 0.25, Z = 0$ and $Z = 3$.

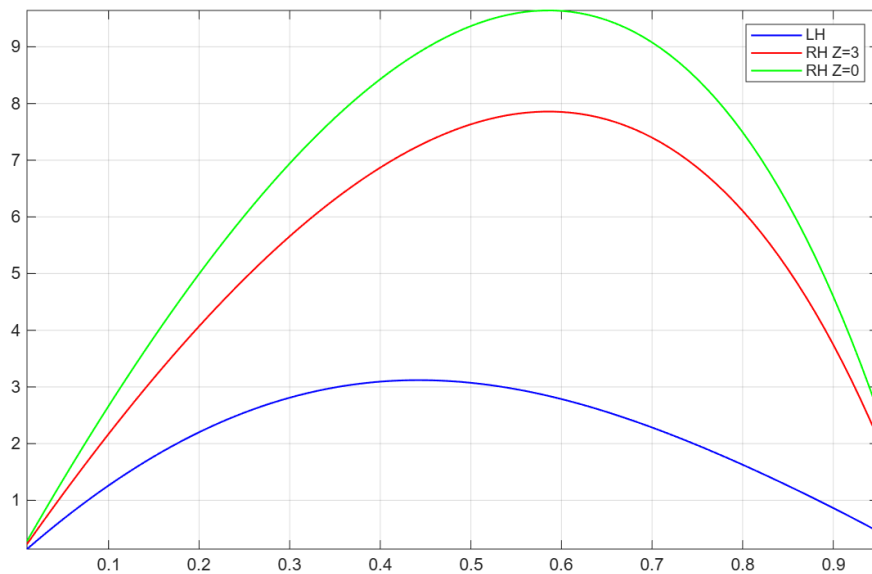


Figure 2: Graphical representation for $\zeta \in [0.01, 0.99]$.

From Figure 2, it is observed that $LH(\zeta) \leq RH_{Z=3}(\zeta) \leq RH_{Z=0}(\zeta)$, which shows that $RH_{Z=3}(\zeta)$ is a sharper error bound than $RH_{Z=0}(\zeta)$. Therefore, for strongly convex functions, we can say that the result of Theorem 2.2 is stronger than that of Corollary 2.3.

3. Applications

3.1. q -digamma function

The q -digamma function ∂_ρ is the ρ -analogue of the digamma function. Under suitable assumptions and applying Theorems 2.2–2.10 with $Z = 0$, one obtains several new inequalities for the q -digamma function (see Propositions 1–5 in the original development).

3.2. Modified Bessel Functions

Consider the function $K_\tau : \mathbb{R} \rightarrow [1, \infty)$ associated with modified Bessel functions as in [30]. Applying Theorems 2.2–2.10 with $Z = 0$ yields a family of Hermite–Hadamard type inequalities for this class of functions (Propositions 6–10).

4. Conclusion

In this paper, new Hermite–Hadamard type inequalities for strongly (s, m) -convex functions have been established by means of Caputo–Fabrizio fractional integrals. The results generalize several classical inequalities and illustrate the advantages of the non-singular kernel. Applications to the q -digamma function and modified Bessel functions further demonstrate the usefulness of the approach.

Future studies could extend these results to different generalized convexity classes, multivariable structures, convex functions in coordinates, and fractional integral operators with different kernel structures such as Atangana–Baleanu, Katugampola, or Prabhakar. It is also thought that similar approaches could lead to new applications in the fields of special functions, optimization theory, and fractional differential equations.

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6. Conflicts of Interest

This work does not have any conflicts of interest.

7. Data Availability Statement

The manuscript contains no original data.

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