

Sharp Coefficient Bounds for a Certain Subclass of Starlike Functions Connected with q -Analogue of the Exponential Function

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Abstract. In this study, by making the use of q -analogue of the exponential function, a new class of starlike functions is defined. The prime contribution of this study covers the derivation of sharp coefficient bounds in open unit disk, especially the first three coefficient bounds, Fekete-Szegő functional and upper bounds of second Hankel determinant for the functions belonging to newly defined class. Moreover, in this study we investigate the Zalcman and generalized Zalcman conjectures of this class functions. In addition, here some known results are discussed based on the choices of the involved parameter q .

1. Introduction and preliminaries

This section will present the fundamental information we will use throughout our study and review existing literature on the topic we are examining.

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. By A , we will denote the class of the functions $f \in H(U)$ given by the following series expansions

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}. \quad (1)$$

It is clear that the function f belonging to the class A satisfies the conditions $f(0) = 0$ and $f'(0) = 1$. The subclass of univalent functions of A is denoted by S . The class S was introduced by Koebe [17] in 1907 and has become the core ingredient of advanced research in this field. Later, many mathematicians became interested in coefficient estimates for this class.

Within a short period, in 1916 Bieberbach [5] published a paper in which the famous coefficient hypothesis was proposed. This hypothesis consists of the following: If $f \in S$ and has the series form (1), then $|a_n| \leq n$ for each $n \geq 2$. In 1985, it was de Branges [8], who settled this long-lasting conjecture. There were a lot of papers devoted to this conjecture in the literature (see [1, 2, 4, 6, 9, 16, 18, 22, 24, 34, 35]).

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It is worth noting that finding bounds for the function coefficients is one of the most fundamental problems in geometric function theory, since it impacts geometric features. For example, the constraint on the second coefficient provides the growth and distortion.

One of the important and extensively studied subclasses of univalent functions is the class of starlike functions. Geometrically, starlike functions are functions that transform the open unit disk U into a starlike region. Starlike function class in the open unit disk U denoted by S^* and defined analytically as follows

$$S^* = \left\{ f \in S : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in U \right\}.$$

In order to obtain better results in the theory of geometric functions, numerous subclasses of the class S have been introduced and studied in recent years as special options for the class S^* (see for example [16, 18, 22, 35, 37, 38]).

One of the important function classes is the Schwarz function class, which we will now define. It is well known that an analytic function ω along the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ is called Schwarz function. This class of functions we will denoted by Ω . For two functions $f, g \in H(U)$, we say that f is subordinate to g (denoted by $f < g$), if there exists a function $\omega \in \Omega$, such that $f(z) = g(\omega(z))$. In particular, if $g \in S$, then

$$f < g \text{ if and only if } f(0) = g(0) \text{ and } f(U) \subset g(U).$$

In 1992 Ma and Minda [20] presented a unified version of the class S^* using subordination terminology. It was defined by

$$S^*(\varphi) = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \varphi(z), z \in U \right\},$$

where $\varphi(z)$ is a univalent function with $\varphi'(0) > 0$ and $\operatorname{Re}(\varphi(z)) > 0$. Additionally, the region $\varphi(U)$ is star-shaped about the point $\varphi(0) = 1$ and is symmetric along the real-line axis. If we choose $\varphi(z) = \frac{1+z}{1-z}$, $z \in U$, then we have a class of starlike functions which is given below:

$$S^* \left(\frac{1+z}{1-z} \right) = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \frac{1+z}{1-z}, z \in U \right\}.$$

Janowski [12] investigated the class of Janowski starlike functions and defined it as follows:

$$S^*(L, M) = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \frac{1 + Lz}{1 + Mz}, z \in U \right\}, \quad -1 \leq M < L \leq 1.$$

Sokol and Stankiewicz [34] chose $\varphi(z) = \sqrt{1+z}$, $z \in U$ and defined the family of class S_L^* as

$$S_L^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \sqrt{1+z}, z \in U \right\}.$$

They examined some geometric properties of the class S_L^* .

Recently, Cho et al. [7] chose $\varphi(z) = 1 + \sin z$, $z \in U$ and defined a class $S_{\sin}^*$ of starlike functions:

$$S_{\sin}^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} < 1 + \sin z, z \in U \right\},$$

and they examined some geometric properties of the class $S_{\sin}^*$.

Bano and Raza [4] chose $\varphi(z) = \cos z$, $z \in U$, and Alotaibi et al. [1] considered $\varphi(z) = \cosh z$, $z \in U$; they defined a family of new classes as follows:

$$S_{\cos}^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \cos z, z \in U \right\} \quad \text{and} \quad S_{\cosh}^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \cosh z, z \in U \right\},$$

and they examined some geometric properties of the classes $S_{\cos}^*$ and $S_{\cosh}^*$.

Recently, Swapur [36] defined a starlike function class associated with the Salagean differential operator and connected with the q -analogue of the hyperbolic tangent function as follows

$$S_s^*(1, q) = \left\{ f \in S : \frac{S_q f(z)}{f(z)} < 1 + \tanh(qz), z \in U \right\}.$$

Swapur found sharp upper bounds for the first four coefficients of functions belonging to this class.

As is known, the study of upper bound estimates of the Fekete-Szegő functional and the Hankel determinant is an important topic in the theory of geometric functions. It is well known that the first order of Hankel determinant of the function $f \in S$ is determined as $H_{2,1}(f) = a_3 - a_2^2$. For $\mu \in \mathbb{C}$ or $\mu \in \mathbb{R}$ the functional $H_{2,1}(f, \mu) = |a_3 - \mu a_2^2|$ is known as the generalized Fekete-Szegő functional [22]. Estimating the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegő problem in the literature. Many researchers have investigated this problem for different subclasses of univalent functions (see [15, 25, 26, 28, 30]).

The Hankel determinants $H_{m,n}(f)$, $m, n \in \mathbb{N}$ for the function $f \in S$ was introduced by Pommerenke [31, 32] and defined by

$$H_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix}, \quad m, n \in \mathbb{N}.$$

It is not hard to note that the second order of Hankel determinant of the function f can be given by $H_{2,2}(f) = a_2 a_4 - a_3^2$.

There are relatively few results about the Hankel determinant for functions belonging to the general family class S . The first sharp inequality for the function $f \in S$ is given by $|H_{2,n}(f)| \leq c \sqrt{n}$, $n \in \mathbb{N}$, where c is a positive constant. This result is due to Hayman [11].

Additionally, for $f \in S$, it is calculated in [30] that $|H_{2,2}(f)| \leq c$, for $c \in [1, \frac{11}{3}]$.

The challenges of determining the bounds of Hankel determinants for a certain set of complex valued functions have attracted the interests of many researchers. For example, Janteng et al. [13, 14] obtained the sharp bounds of $|H_{2,2}(f)|$ for the subfamilies S^* and C , where C is convex function class in $\mathcal{U}$. They give the following estimates

$$|H_{2,2}(f)| \leq \begin{cases} 1 & \text{for } f \in S^*, \\ 1 & \text{for } f \in C. \end{cases}$$

In 1960, Zalcman defined the conjecture for univalent functions. He stated that every $f \in S$ of the form (1) satisfies the following inequality:

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad (2)$$

for $n \in \mathbb{N}_2 = \{2, 3, \dots\}$.

In 1999, Ma [21] proved a generalized version of Zalcman conjecture and stated that every univalent function $f \in S$ satisfies the following inequality:

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \quad (3)$$

for $n, m \in \mathbb{N}_2 = \{2, 3, \dots\}$.

For $n = m = 2$, the inequality (2) and (3) has the form

$$|a_2^2 - a_3| \leq 1.$$

For $q \in (0, 1)$ the q -analogue of the exponential function e^{qz} defined as follows

$$e^{qz} = \sum_{n=0}^{\infty} \frac{(qz)^n}{n!} = 1 + \frac{qz}{1!} + \frac{(qz)^2}{2!} + \frac{(qz)^3}{3!} + \cdots + \frac{(qz)^n}{n!} + \cdots, \quad (4)$$

It is clear that $\lim_{q \rightarrow 1^-} e^{qz} = e^z$.

By using the definition of subordination, we introduce a new subclass of starlike functions connected to q -analogue of the exponential function.

Definition 1.1. For $q \in (0, 1)$ a function $f \in S$ is said to be in the class $S^*(e^{qz})$ if it satisfies the following condition

$$\frac{zf'(z)}{f(z)} < e^{qz}, \quad z \in U; \quad \text{that is, } S^*(e^{qz}) = \left\{ f \in S : \frac{zf'(z)}{f(z)} < e^{qz}, \quad z \in U \right\}.$$

In the case $q \rightarrow 1^-$, from Definition 1.1 we obtain the following class of functions.

Definition 1.2. A function $f \in S$ is said to be in the class $S^*(e^z)$ if it satisfies the following condition

$$\frac{zf'(z)}{f(z)} < e^z, \quad z \in U; \quad \text{that is, } S^*(e^z) = \left\{ f \in S : \frac{zf'(z)}{f(z)} < e^z, \quad z \in U \right\}.$$

Remark 1.3. It is worth noting that the classes defined above are not empty.

Indeed, for $q \in (0, 1)$ the function

$$f(z) = z + qz^2 + \frac{3q^2}{4}z^3 + \dots$$

is a member of the class $S^*(e^{qz})$, which derived from the equality

$$\frac{zf'(z)}{f(z)} = e^{qz}, \quad z \in U.$$

This equality is a consequence of the condition in the definition of the class $S^*(e^{qz})$.

Also, in the special case $q \rightarrow 1^-$, the function

$$f(z) = z + z^2 + \frac{3}{4}z^3 + \dots$$

is a member of the class $S^*(e^z)$.

Let's define another class of functions that we will frequently use throughout our study.

Let P be the class of analytic functions in U with $\operatorname{Re}(p(z)) > 0$ and $p(0) = 1$. This class can be defined using the subordination principle as follows:

$$P = \left\{ p \in A : p(z) < \frac{1+z}{1-z}, \quad z \in U \right\}.$$

It is clear that the functions that satisfy these conditions have the following series expansion

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots = 1 + \sum_{n=1}^{\infty} p_nz^n, \quad z \in U. \quad (5)$$

The class P defined above is known as the class Carathéodory functions in the literature [23].

The subsequent lemmas are essential for the proof our main results.

It should be noted that the equality given for p_2 in Lemma 1.1 below was proven by Pommerenke [32] and the equality given for p_3 was proven by Libera and Złotkiewicz [19].

Lemma 1.4. Let the an analytic function p be of the form (5), then

$$2p_2 = p_1^2 + (4 - p_1^2)x, \quad (6)$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)y, \quad (7)$$

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Lemma 1.5. [10]. Let the function p belong to the class P . Then,

$$|p_n| \leq 2 \quad \text{for each } n \in \mathbb{N}. \quad (8)$$

Also,

$$|p_n - \lambda p_k p_{n-k}| \leq 2 \quad \text{for } n, k \in \mathbb{N}, n > k \text{ and } \lambda \in [0, 1]. \quad (9)$$

The equality hold for the function $p(z) = (1+z)/(1-z)$.

Lemma 1.6. [3]. Let the function p belong to the class P . Then,

$$|p_3 - 2Bp_1p_2 + Dp_1^3| \leq 2, \quad \text{if } 0 \leq B \leq 1 \text{ and } B(2B-1) \leq D \leq B.$$

In this study, we defined the sharp coefficient bounds on the open unit disk, specifically the first three coefficient bounds, the upper bounds of the Fekete-Szegő functional, and the upper bounds of the second Hankel determinant for functions belonging to the newly defined class $S^*(e^{qz})$. Additionally, in the study we examine the Zalcman conjecture for the defined class. Moreover, some known results are highlighted based on the choice of the parameter q .

2. Main Results

2.1. Upper Bound Estimate for the Coefficients

In this subsection, we give some coefficient estimates for the functions belonging to the class $S^*(e^{qz})$. Let's give the following theorem on upper bound estimates of coefficients.

Theorem 2.1. Let the function f given by series expansions (1) belong to the class $S^*(e^{qz})$. Then, we have the following estimates

$$|a_2| \leq q, \quad |a_3| \leq \frac{q}{4} \begin{cases} 2 & \text{if } q \in (0, \frac{2}{3}] \\ 3q & \text{if } q \in [\frac{2}{3}, 1) \end{cases}, \quad \text{and} \quad |a_4| \leq \frac{(3q+2)q}{6}.$$

Obtained here results are sharp.

Proof. Let $f \in S^*(e^{qz})$, then exists Schwarz function $\omega : U \rightarrow U$, such that

$$\frac{zf'(z)}{f(z)} = e^{q\omega(z)}, \quad z \in U. \quad (10)$$

According to the relationship between the Schwarz and Carathéodory function classes, there exists a function $p \in P$ such that

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} \Leftrightarrow \omega(z) = \frac{p(z) - 1}{p(z) + 1}.$$

It is follows from that

$$\omega(z) = \frac{p_1}{2}z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 + \dots, \quad z \in U. \quad (11)$$

If we consider equality (11) in the expansion (4), we can write

$$\begin{aligned} e^{q\omega(z)} &= 1 + \frac{q}{2}p_1z + \frac{q}{2}\left(p_2 - \frac{2-q}{4}p_1^2\right)z^2 \\ &\quad + \frac{q}{2}\left(p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2-6q+6}{24}p_1^3\right)z^3 + \dots, \quad z \in U. \end{aligned} \quad (12)$$

Using simple differentiation operations, we write the following equality

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \dots, \quad z \in U. \quad (13)$$

Taking into account the expansions of (12) and (13) in the equality (10), and equating the coefficients of the same powers with respect to the variable z in the resulting equality, we obtain:

$$a_2 = \frac{q}{2}p_1, \quad (14)$$

$$2a_3 - a_2^2 = \frac{q}{2} \left(p_2 - \frac{2-q}{4}p_1^2 \right), \quad (15)$$

$$3a_4 - 3a_2a_3 + a_2^3 = \frac{q}{2} \left(p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2 - 6q + 6}{24}p_1^3 \right). \quad (16)$$

Using inequality (8) (see Lemma 1.5) to the equality (14), we have the first result of theorem. Now let's find an upper bound for the modulus of the coefficient a_3 .

Since $p_2 = \frac{p_1^2}{2} + \frac{4-p_1^2}{2}x$ for some $x \in \mathbb{C}$ with $|x| \leq 1$ (see Lemma 1.4), then from (15) can written

$$a_3 = \frac{q}{16} [2(4 - p_1^2)x + 3qp_1^2]. \quad (17)$$

Using triangle inequality to the equality (17), we can write the following inequality:

$$|a_3| \leq \frac{q}{16} [2(4 - t^2)\xi + 3qt^2], \quad (18)$$

where, $\xi = |x|$ and $t = |p_1|$.

Let's define the function $\chi(\xi) = 2(4 - t^2)\xi + 3qt^2$, $\xi \in [0, 1]$. Since $\chi'(\xi) = 4 - t^2 \geq 0$, $\xi \in [0, 1]$, then from (18), we write

$$|a_3| \leq \frac{q}{16} [(3q - 2)t^2 + 8].$$

Maximizing the function $\phi : [0, 2] \rightarrow \mathbb{R}$ defined as follows

$$\phi(t) = (3q - 2)t^2 + 8, \quad t \in [0, 2],$$

we can easily see that $\phi(t) \leq 8$ if $q \in (0, \frac{2}{3}]$ and $\phi(t) \leq 12q$ if $q \in [\frac{2}{3}, 1)$.

So that

$$|a_3| \leq \frac{q}{4} \begin{cases} 2 & \text{if } q \in (0, \frac{2}{3}] \\ 3q & \text{if } q \in [\frac{2}{3}, 1) \end{cases}.$$

Now let's find an upper bound for the modulus of the coefficient a_4 . From (16), we get

$$a_4 = a_2 \left(a_3 - \frac{1}{3}a_2^2 \right) + \frac{q}{6} \left(p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2 - 6q + 6}{24}p_1^3 \right).$$

Using triangle inequality to the last equality we have

$$|a_4| \leq |a_2| \left| a_3 - \frac{1}{3}a_2^2 \right| + \frac{q}{6} \left| p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2 - 6q + 6}{24}p_1^3 \right|. \quad (19)$$

Then, from (15) we can write

$$a_3 - \frac{1}{3}a_2^2 = \frac{q}{4} \left(p_2 - \frac{6-5q}{12}p_1^2 \right).$$

Since $(6 - 5q)/12 \in [0, 1]$ for $q \in (0, 1)$, then using inequality (9) (see Lemma 1.5) to the last equality, we have

$$\left| a_3 - \frac{1}{3}a_2^2 \right| \leq \frac{q}{2}. \quad (20)$$

Now let's find an upper bound evaluation for

$$\left| p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2-6q+6}{24}p_1^3 \right|.$$

For this, we write

$$p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2-6q+6}{24}p_1^3 = p_3 - 2Bp_1p_2 + Dp_1^3,$$

where $B = (2 - q)/4 \in [0, 1]$ and $D = (q^2 - 6q + 6)/24$.

Since $B = (2 - q)/4 \in [0, 1]$ and $B(2B - 1) \leq D \leq B$ for $q \in (0, 1)$, then

$$\left| p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2-6q+6}{24}p_1^3 \right| \leq 2 \quad (21)$$

(see Lemma 1.6).

Finally, taking into account inequalities (21), (20) and the evaluation we found for a_2 , from the inequality (19) we arrive at the following inequality:

$$|a_4| \leq \frac{(3q+2)q}{6}.$$

For sharpness of the first result, consider the function $f : U \rightarrow \mathbb{C}$, defined by

$$\frac{zf'(z)}{f(z)} = e^{qz} \Rightarrow f(z) = z + qz^2 + \frac{3q^2}{4}z^3 + \dots$$

Also, in the case $q \in [\frac{2}{3}, 1)$ second result of the theorem is sharp for this above function.

Third result of the theorem is sharp for the function

$$f(z) = z + \frac{(3q+2)q}{6}z^4 + \dots$$

Thus, the proof of theorem is completed. $\square$

In the special case $q \rightarrow 1^-$, from Theorem 2.1 we have the following result for the class $S^*(e^z)$.

Corollary 2.2. *If $f \in S^*(e^z)$, then we have*

$$|a_2| \leq 1, \quad |a_3| \leq \frac{3}{4} \quad \text{and} \quad |a_4| \leq \frac{5}{6}.$$

2.2. Fekete-Szegő Problem

In this subsection, we will examine the Fekete-Szegő problem for the class $S^*(e^{qz})$.

Theorem 2.3. *Let $f \in S^*(e^{qz})$ and $\mu \in \mathbb{C}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{q}{4} \begin{cases} 2 & \text{if } |3 - 4\mu| \leq 2/q, \\ |3 - 4\mu|q & \text{if } |3 - 4\mu| \geq 2/q. \end{cases}$$

Proof. Let $f \in S^*(e^{qz})$ and $\mu \in \mathbb{C}$. Then, from the equalities (14) and (15) the generalized Fekete-Szegő functional $a_3 - \mu a_2^2$ can be written as follows

$$a_3 - \mu a_2^2 = \frac{1-2\mu}{2} a_2^2 + \frac{q}{4} \left(p_2 - \frac{2-q}{4} p_1^2 \right).$$

Applying the equality (6) (see Lemma 1.4) to the last equality and considering (14), we get

$$a_3 - \mu a_2^2 = \frac{q}{16} \left[(3-4\mu)qp_1^2 + 2(4-p_1^2)x \right] \quad (22)$$

for some $x \in \mathbb{C}$ with $|x| \leq 1$.

Using triangle inequality to the equality (22), we have

$$|a_3 - \mu a_2^2| \leq \frac{q}{16} \left[|3-4\mu|qt^2 + 2(4-t^2)\xi \right] \quad (23)$$

where, $\xi = |x|$ and $t = |p_1|$.

By maximizing the expression on the right side of the inequality (23) with respect to the variable $\xi \in [0, 1]$, we obtain the following inequality

$$|a_3 - \mu a_2^2| \leq \frac{q}{16} \left[(|3-4\mu|q-2)t^2 + 8 \right], \quad t \in [0, 2]. \quad (24)$$

Maximizing the function $\phi : [0, 2] \rightarrow \mathbb{R}$ defined as follows

$$\phi(t) = (|3-4\mu|q-2)t^2 + 8, \quad t \in [0, 2],$$

we can easily see that $\phi(t) \leq 8$ if $|3-4\mu| \leq 2/q$ and $\phi(t) \leq 4|3-4\mu|q$ if $|3-4\mu| \geq 2/q$. When we take these into account in the inequality (24), we arrive at the desired result of theorem.

Thus, proved the theorem. $\square$

In the special cases $\mu = 0$ and $\mu = 1$, from Theorem 2.3 we obtain the following results, respectively.

Corollary 2.4. If $f \in S^*(e^{qz})$, then

$$|a_3| \leq \frac{q}{4} \begin{cases} 2 & \text{if } q \in \left(0, \frac{2}{3}\right], \\ 3q & \text{if } q \in \left[\frac{2}{3}, 1\right). \end{cases}$$

Corollary 2.5. If $f \in S^*(e^{qz})$, then

$$|a_3 - a_2^2| \leq \frac{q}{2}.$$

Remark 2.6. As can be seen that the estimate obtained in Corollary 2.4 confirms the second result of Theorem 2.1.

In addition, when $q \rightarrow 1^-$ from Theorem 2.3, we have the following result.

Corollary 2.7. If $f \in S^*(e^z)$ and $\mu \in \mathbb{C}$, then we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \begin{cases} 2 & \text{if } |3-4\mu| \leq 2, \\ |3-4\mu| & \text{if } |3-4\mu| \geq 2. \end{cases}$$

Furthermore, in the special cases $\mu = 0$ and $\mu = 1$, from Corollary 2.7 we obtain the following results, respectively.

Corollary 2.8. If $f \in S^*(e^z)$, then we have

$$|a_3| \leq \frac{3}{4}.$$

Corollary 2.9. If $f \in S^*(e^z)$, then we have

$$|a_3 - a_2^2| \leq \frac{1}{2}.$$

2.3. Upper Bound Estimate for the Second Hankel Determinant

Now let's give an upper bound estimate of the second Hankel determinant for the functions belonging to the class $S^*(e^{qz})$.

Theorem 2.10. *Let the function f given by series expansions (1) belong to the class $S^*(e^{qz})$. Then, we have*

$$|a_2a_4 - a_3^2| \leq \frac{q^2}{24} \begin{cases} 8q^2 + 14 & \text{if } q \in (0, \frac{2}{3}] \\ 8q^2 + 9q + 8 & \text{if } q \in [\frac{2}{3}, 1] \end{cases}.$$

Proof. To prove Theorem 2.10, we will use some results obtained in Subsection 2.1 and Subsection 2.2.

Let $f \in S^*(e^{qz})$, then find the expression $a_2a_4 - a_3^2$ from (16) and apply the triangle inequality, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{|a_2|^4}{3} + |a_2||a_3 - a_2^2| + \frac{q}{6}|a_2| \left| p_3 - \frac{2-q}{2}p_1p_2 + \frac{q^2 - 6q + 6}{24}p_1^3 \right|. \quad (25)$$

Taking into account inequality (21) and the estimates we obtained for $|a_2|$, $|a_3|$ and $|a_3 - a_2^2|$ in Subsection 2.1 and Corollary 2.5, respectively in inequality (25), we arrive at the following result:

$$|a_2a_4 - a_3^2| \leq \frac{q^2}{24} \begin{cases} 8q^2 + 14 & \text{if } q \in (0, \frac{2}{3}] \\ 8q^2 + 9q + 8 & \text{if } q \in [\frac{2}{3}, 1] \end{cases}.$$

Thus, the proof of theorem is completed. $\square$

In the case $q \rightarrow 1^-$ from the Theorem 2.10, we have the following result.

Corollary 2.11. *If $f \in S^*(e^z)$, then we have*

$$|a_2a_4 - a_3^2| \leq \frac{25}{24}.$$

2.4. Zalcman and Generalized Zalcman Conjecture

In this subsection, we will examine the Zalcman conjecture for the class $S^*(e^{qz})$.

As we stated in the introduction to our study, Zalcman in the assumption she defined in 1960, states that every univalent function $f \in S$ satisfies the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2$$

for $n \in \mathbb{N}_2 = \{2, 3, \dots\}$.

In 1999, Ma [21] proved a generalized version of Zalcman conjecture and stated that every univalent function $f \in S$ satisfies the following inequality

$$|a_na_m - a_{n+m-1}| \leq (n-1)(m-1)$$

for $n, m \in \mathbb{N}_2 = \{2, 3, \dots\}$.

For $n = m = 2$, the above inequalities has the form

$$|a_2^2 - a_3| \leq 1.$$

In Subsection 2.2 (see Corollary 2.5) of our study, we proved the following result.

Theorem 2.12. *Let the function f given by series expansions (1) belong to the class $S^*(e^{qz})$. Then, we have*

$$|a_2^2 - a_3| \leq \frac{q}{2}.$$

Bound obtained here is sharp for the function $f(z) = z + \frac{q}{2}z^3 + \dots$, $z \in U$.

When $q \rightarrow 1^-$, from Theorem 2.12, we have the following result.

Corollary 2.13. *If $f \in S^*(e^z)$, then we have*

$$|a_2^2 - a_3| \leq \frac{1}{2}.$$

Now let's give the following theorem on the generalized Zalcman conjecture.

Theorem 2.14. *Let the function f given by series expansions (1) belong to the class $S^*(e^{qz})$. Then, we have*

$$|a_2a_3 - a_4| \leq \frac{(q^2 + 1)q}{3}.$$

Bound obtained here is sharp.

Proof. Let $f \in S^*(e^{qz})$, then from the equality (16) we can write

$$a_4 - a_2a_3 = \frac{q}{6} \left(p_3 - \frac{2-q}{2} p_1p_2 + \frac{q^2 - 6q + 6}{24} p_1^3 \right) - \frac{a_2^3}{3}.$$

Applying the triangle inequality to this equality and using inequalities (21) and $|a_2| \leq q$, we arrive at the result of theorem.

Obtained result in Theorem 2.14 is sharp for the function

$$f(z) = z + \frac{(q^2 + 1)q}{3} z^4 + \dots, \quad z \in U.$$

□

When $q \rightarrow 1^-$, from Theorem 2.14, we have the following result.

Corollary 2.15. *If $f \in S^*(e^z)$, then we have*

$$|a_4 - a_2a_3| \leq \frac{2}{3}.$$

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