TURKISH JOURNAL OF SCIENCE
VOLUME 11, ISSUE 2, 58–64
ISSN: 2587–0971
https://engineering-journals.org

Multiplicative dependence between Padovan and Narayana numbers

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Abstract. The Padovan sequence and the Narayana sequence represent two classical instances of third-order linear recurrence streams, which are established via the respective structural relations $P_n = P_{n-2} + P_{n-3}$ with the starting configuration $(P_0, P_1, P_2) = (1, 1, 1)$, and $N_m = N_{m-1} + N_{m-3}$ under the primary values $(N_0, N_1, N_2) = (0, 1, 1)$. This paper is dedicated to providing a comprehensive analysis regarding the multiplicative dependency between these two separate recursive structures.

1. Introduction

The algebraic properties of sequences generated by linear recurrence relations have long captured the attention of researchers in number theory. In particular, the study of arithmetic intersections and functional dependencies between different recurrence streams represents a classic line of investigation. Two non-zero integers a and b are classified as multiplicatively dependent if there exist integers x and y, not simultaneously zero, such that the exponential identity $a^x = b^y$ is satisfied. Conversely, when no such nontrivial integer combinations exist, the numbers are designated as multiplicatively independent.

A significant amount of literature has been dedicated to identifying the precise instances of multiplicative dependence within single linear recurrences or across distinct families of sequences. For example, pioneering frameworks completely mapped these interactions for classical sequences such as the Fibonacci, Lucas, Padovan, Perrin and generalized Tribonacci numbers [1–10]. Most of these breakthroughs were achieved by combining the deep theoretical machinery of linear forms in logarithms with algorithmic reduction methods designed to handle large computational boundaries.

To analyze the asymptotic growth and algebraic behavior of the structures under our consideration, we recall their underlying characteristic equations. The recurrence stream of the Padovan sequence $\{P_n\}_{n\geq 0}$ is governed by the polynomial equation $x^3 - x - 1 = 0$, which possesses a unique real dominant root $\alpha \approx 1.32471$, commonly referred to as the plastic ratio. On the other hand, the Narayana sequence $\{N_m\}_{m\geq 0}$ is characterized by the algebraic equation $x^3 - x^2 - 1 = 0$, whose single real dominant root is denoted as $\alpha_N \approx 1.46557$. For sufficiently large indices, both sequences can be explicitly parameterized via their analytic Binet-like expansion forms:

$$P_n = a\alpha^n + b\beta^n + c\gamma^n$$

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Received: 17 August 2026; Accepted: 07 September 2026; Published: 13 October 2026

Keywords. Padovan sequences, Narayana numbers, Logarithmic linear forms, LLL reduction technique, Multiplicative dependence

2010 *Mathematics Subject Classification.* 11B39, 11B37, 11J86

Cited this article as: Çağman, A., Multiplicative dependence between Padovan and Narayana numbers, Turkish Journal of Science, 11(2), 2026, 58–64

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and

$$N_m = a_N\alpha_N^m + b_N\beta_N^m + c_N\gamma_N^m$$

where β, γ and β_N, γ_N represent the remaining complex conjugate roots of the respective characteristic polynomials. The primary structural constants associated with the dominant terms are explicitly evaluated in their simplest forms as:

$$a = \frac{1}{3\alpha^2 - 1} \qquad \text{and} \qquad a_N = \frac{\alpha_N^3}{\alpha_N^3 + 2}.$$

Since the complex roots lie strictly inside the unit disk in the complex plane, their contributions $e_1(n) = P_n - a\alpha^n$ and $e_2(m) = N_m - a_N\alpha_N^m$ vanish as the indices tend toward infinity, leaving $a\alpha^n$ and $a_N\alpha_N^m$ as the dominant asymptotic approximations.

In the present paper, our primary objective is to shift this focus toward the algebraic interplay between the Padovan and Narayana structures. Specifically, we aim to provide a rigorous and comprehensive description of all integer solutions for the exponential Diophantine equation:

$$P_n^x = N_m^y \tag{1.1}$$

where the exponents x, y, n and m are non-negative integers.

Considering the early boundary values of both sequences, we observe that $P_0 = P_1 = P_2 = 1$ and $N_1 = N_2 = N_3 = 1$. Consequently, equation (1.1) yields a set of trivial identities for any arbitrary non-zero choices of x and y whenever the indices fall into the sets $n \in \{0, 1, 2\}$ and $m \in \{1, 2, 3\}$. To ensure the analysis deals strictly with non-trivial algebraic structures, we impose the constraints $n \geq 3$ and $m \geq 4$.

The central findings of this investigation are formalized in the following main statement.

Theorem 1.1. *The Diophantine equation* (1.1) *admits solutions in positive indices* $x, y, n \geq 3$, $m \geq 4$ *if and only if the index pairs* (n, m) *belong to the finite configuration:* $(n, m) \in \{(3, 4), (5, 5), (6, 6), (9, 8)\}$ *with* $x = y$, $\{(3, 6), (4, 6), (5, 8)\}$ *with* $x = 2y$, $\{(6, 4), (9, 5), (11, 6)\}$ *with* $y = 2x$ *and* $\{(11, 4)\}$ *with* $y = 4x$.

2. Materials and Methods

To establish the analytical framework required for the proof of our main result, we must first recall several fundamental arithmetic definitions and auxiliary results. This section serves to compile the essential properties of algebraic heights, the foundational theorem on linear forms in logarithms, and the algorithmic lattice reduction machinery.

We begin by formalizing the concept of the logarithmic Weil height, which acts as a crucial measurement of the arithmetic complexity of an algebraic number. Let γ be an algebraic number of degree d over the rational field $\mathbb{Q}$, whose minimal primitive polynomial over the integers $\mathbb{Z}$ is given by

$$P(x) = a_0 \prod_{i=1}^{d} (x - \gamma^{(i)})$$

where the leading coefficient a_0 is a strictly positive integer and $\gamma^{(i)}$'s are conjugates of γ. The absolute logarithmic Weil height of γ is explicitly defined as:

$$h(\gamma) = \frac{1}{d}\left(\log(a_0) + \sum_{i=1}^{d} \log\Big(\max\{|\gamma^{(i)}|, 1\}\Big)\right).$$

In the specific scenario where γ simplifies to a rational fraction p/q in its lowest terms with $q > 0$, this valuation naturally collapses to:

$$h(p/q) = \log(\max\{|p|, q\}).$$

For any pair of algebraic numbers η and γ, the height function demonstrates well-known behaviors under standard algebraic operations:

- $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log(2)$
- $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$
- $h(\eta^t) = |t|\, h(\eta)$ for any rational exponent $t \in \mathbb{Q}$.

Utilizing these structural properties alongside the specific algebraic characteristics of our recurrence dynamics, we deduce the explicit height valuations for the dominant roots and operational coefficients:

$$h(\alpha) = \frac{\log \alpha}{3}, \quad h(\alpha_N) = \frac{\log \alpha_N}{3}, \quad h(a) = \frac{\log 23}{3}, \quad h(P_n) = \log(P_n), \quad h(N_m) = \log(N_m).$$

A pivotal component of our bounding technique involves obtaining a precise lower restriction for non-vanishing linear combinations of logarithms. To achieve this, we invoke a celebrated deep result due to Matveev, which provides a powerful lower bound for homogeneous rational linear forms.

Theorem 2.1 (Matveev, 2000). *Let $\gamma_1, \gamma_2, \ldots, \gamma_t$ be real algebraic numbers belonging to a valid algebraic number field K of degree D over $\mathbb{Q}$. Let $b_1, b_2, \ldots, b_t$ be rational integers, and define the algebraic logarithmic form as:*

$$\Lambda := \gamma_1^{b_1}\gamma_2^{b_2}\cdots\gamma_t^{b_t} - 1.$$

Let $B \geq \max\{|b_1|, |b_2|, \ldots, |b_t|\}$ represent an explicit upper bound for the absolute values of the rational multipliers. Furthermore, let A_i be real parameters chosen such that

$$A_i \geq \max\{Dh(\gamma_i), |\log(\gamma_i)|, 0.16\}$$

for every index $i = 1, \ldots, t$. Under the essential assumption that Λ is non-zero, the following inequality holds:

$$|\Lambda| > \exp\Big(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 \cdot (1 + \log(D)) \cdot (1 + \log(B)) \cdot A_1 A_2 \cdots A_t \Big).$$

The theoretical application of Matveev's framework inevitably yields immense initial boundaries for our index variables, which are far too large for brute-force verification. To bridge this gap and reduce these constraints to computationally tractable levels, we employ an algorithmic mechanism rooted in the geometry of numbers—specifically, the Lenstra–Lenstra–Lovász (LLL) lattice reduction technique. The exact mathematical formulation we utilize is summarized in the following lemma. It can be found in Proposition 2.3.20 of [11].

Lemma 2.2. *Let $X_1, X_2, \ldots, X_t$ be positive integer bounds such that the integer variables satisfy $|x_i| \leq X_i$, and let $X := \max\{X_i\}$. Let C be a large fixed scaling constant chosen to comply with $C > (t \cdot X)^t$. Consider the integer lattice Ω embedded in $\mathbb{R}^t$, which is explicitly spanned by the coordinate basis vectors:*

$$b_j = e_j + [C \cdot \alpha_j] \cdot e_t \quad \text{for } 1 \leq j \leq t-1, \qquad b_t = [C \cdot \alpha_t] \cdot e_t$$

where $\{e_1, e_2, \ldots, e_t\}$ denotes the standard canonical base of the Euclidean space $\mathbb{R}^t$, and the bracket $[x]$ denotes the nearest integer to x. Let $\{b_i\}$ be a reduced basis of the lattice Ω obtained via the LLL scheme, and let $\{b_i^*\}$ represent its associated orthogonal Gram–Schmidt sequence. Let

$$c_1 = \max_{1 \leq i \leq t} \frac{\|b_1\|}{\|b_i^*\|}, \qquad \delta = \frac{\|b_1\|}{c_1}, \qquad Q = \sum_{i=1}^{t-1} X_i^2, \qquad T = \sum_{i=1}^{t} X_i^2.$$

If the geometric condition $\delta^2 \geq T^2 + Q$ is fully satisfied, then the linear form obeys the lower estimate:

$$\left| \sum_{i=1}^{t} x_i \cdot \alpha_i \right| \geq \frac{\sqrt{\{\delta^2 - Q\}} - T}{C}.$$

Finally, to derive precise upper bounds from our transcendental inequalities, we employ a well-known analytical lemma tracking back to the work of [12].

Lemma 2.3. *Let $m \geq 1$ be an integer. If the real numbers T and x satisfy the functional conditions $T > (4 \cdot m^2)^m$ and $T > x/(\log(x))^m$, then the variable x is bounded above by:*

$$x < 2^m \cdot T \cdot (\log(T))^m.$$

3. Proof of Theorem 1.1

Let us first screen the lower boundary index options for the equation (1.1). For the index ranges $n = 0, 1, 2$ and $m = 1, 2, 3$, both sequences hold values identical to 1. Consequently, any combinations chosen within these initial steps yield trivial representations of the identity $1^x = 1^y$. To avoid these uninformative states, we proceed under the strict constraints $n \geq 3$ and $m \geq 4$. Furthermore, we can assume without any loss of generality that the exponents x and y are relatively prime, meaning $\gcd(x, y) = 1$. Let $l := \min\{n, m\}$ represent the minimum index marker and let $L := \max\{n, m + 1\}$ serve as the upper bound dictated by the growth rate of the Narayana component. By matching the asymptotic growth of both sequences through their common base representation, we establish the explicit linear constraints $y \leq n$ and $x \leq m + 1$, which limits the maximum exponent magnitude to $B := \max\{x, nx, y\} \leq n(m + 1)$.

To explicitly analyze the transcendental nature of our exponential Diophantine equation, we must establish precise linear forms in logarithms. By expanding equation (1.1) via the dominant terms of the Binet-like formulas and controlling the remaining small algebraic perturbations, we arrive at our first approximation. Specifically, the first linear form, which isolates the asymptotic behavior of the Padovan component, satisfies the following inequality:

$$|\Delta_1| := \left|N_m^y a^{-x} \alpha^{-nx} - 1\right| < \frac{2(m + 1)}{(1.32)^n} \tag{3.1}$$

In a fully symmetric manner, we can capture the alternative growth trajectory by focusing on the dominant root of the Narayana sequence. This technical step yields a companion inequality that governs the alternate index growth:

$$|\Delta_2| := \left|P_n^x a_N^{-y} \alpha_N^{-my} - 1\right| < \frac{2n}{(1.46)^m} \tag{3.2}$$

To apply Matveev's deep theoretical reduction machinery, it is essential to bridge these two independent linear relations into a unified cross-bounding framework. By combining the algebraic properties of $|\Delta_1|$ and $|\Delta_2|$ through a simultaneous logarithmic expansion, we can isolate the joint growth dynamics. Under the structural assumption that $l = \min\{n, m\}$, this combination yields the final transitional linear form in logarithms:

$$|\Delta_3| := \left|\frac{a^x}{a_N^y} \alpha^{nx} \alpha_N^{-my} - 1\right| < \frac{2.4 \times 10^{15}}{1.32^l} \tag{3.3}$$

This sharp inequality effectively connects the independent variables of both sequences.

At this stage, lower bounds for the three linear forms are obtained by applying Matveev's theorem (Theorem 2.1) as follows:

$$n < 2.36 \times 10^{13} \log(L)(m + 1) \tag{3.4}$$

$$m < 4.25 \times 10^{11} \log(L)(n - 1) \tag{3.5}$$

and

$$L < 2.36 \times 10^{13} \cdot \log(L) \cdot l \tag{3.6}$$

$$l < 1.54 \times 10^{12} \log(L). \tag{3.7}$$

Consequently, by applying Lemma 2.2 to the transcendental inequality, we explicitly establish the absolute numerical upper bound for the system as $L < 4.92 \times 10^{29}$.

The theoretical numerical upper bound for L is far too large to allow direct brute-force computer search. As a result, we now deploy the Lenstra–Lenstra–Lovász (LLL) lattice basis reduction algorithm to significantly decrease the upper bound for L.

Assume that $l > 255$. Denoting the non-zero linear form in logarithms as $\Gamma_3 := x \log(a/a_N) + nx \log(\alpha) - my \log(\alpha_N)$, it can be observed that $\Gamma_3 \neq 0$ because $\Delta_3 = e^{\Gamma_3} - 1 \neq 0$. From our cross-indexed estimation and

using the global bound for L, we can notice that:

$$|\Delta_3| = |e^{\Gamma_3} - 1| < \frac{1}{2}.$$

Thus, by applying the basic analytic mapping $e^{|\Gamma_3|} < \frac{3}{2}$, we can bound the logarithmic form directly. From inequality (3.3), we have:

$$|\Gamma_3| < e^{|\Gamma_3|}|e^{\Gamma_3} - 1| < \frac{30L}{(1.32)^l} \tag{3.8}$$

To find a concrete lower bound for Γ_3, consider the two-dimensional lattice Ω_3 generated by the coordinate columns of the matrix approximation mapping the linear combination of the logarithms. Here, we identify the operational parameters as follows:

$$X_1 = x, \quad X_2 = y, \quad X_3 = |nx\log(\alpha) - my\log(\alpha_N)|,$$

with the corresponding log-parameters $\alpha_1 = \log(a)$, $\alpha_2 = \log(\alpha_N)$. We set $X = L^2$ as an initial upper bound of X_1, X_2 and choose a large scaling constant $C = 10^{120} > (2X)^2$. Using the LLL algorithm implemented in a computer algebra system, we find a reduced base $\{b_i\}$ for the lattice Ω_3 and compute the associated Gram–Schmidt orthogonalized base $\{b_i^*\}$. We also calculate the tracking constants c_1, δ, Q, and T. By applying Lemma 2.1, we obtain the verified lower bound:

$$|\Gamma_3| \geq 1.14 \times 10^{-65}.$$

Combining this computational lower inequality with the geometric decay inequality established in (3.8), and performing a sequence of direct algebraic simplifications, we get the remarkably smaller bound:

$$l \leq 782.$$

Thus, by substituting this minimized index back into the unified upper bound relation in (3.6), we get:

$$\frac{L}{\log L} < 2.36 \times 10^{13} \times 782.$$

By applying Lemma 2.2 once again to the refined logarithmic expression under the initial reduction state $l \leq 782$, we successfully achieve a drastically tighter intermediate bound for the global index variable:

$$L < 1.94 \times 10^{18}. \tag{3.9}$$

Further, we introduce the auxiliary companion linear forms in logarithms to check the remaining index directions, defined as $\Gamma_1 := y\log(N_m) - x\log(a) - nx\log(\alpha)$ and $\Gamma_2 := x\log(P_n) - y\log(a_N) - my\log(\alpha_N)$. It can be verified that each $\Gamma_i \neq 0$ since their exponential counterparts satisfy $\Delta_i = e^{\Gamma_i} - 1 \neq 0$. Utilizing the initial estimates from equations (3.1) and (3.2) along with the updated ceiling in (3.9), for $i = 1$ or 2, we obtain:

$$|\Gamma_i| < e^{|\Gamma_i|}|e^{\Gamma_i} - 1| < \frac{10^{20}}{1.32^{n_i}} \tag{3.10}$$

with $n_1 = n$ and $n_2 = m$, where $|\Delta_i| < \frac{1}{2}$ and $e^{|\Gamma_i|} < \frac{3}{2}$ are rigorously satisfied.

As $l \leq 782$, two main sub-cases naturally arise: either $n \in [3, 782]$ or $m \in [0, 782]$. For the first case where $n \in [3, 782]$, we aim to calculate a precise lower bound for Γ_2 via the LLL algorithm. For this purpose, we set the matrix scaling boundaries as $X = (1.94 \times 10^{18})^2$ as an absolute upper bound for $X_1 = x, X_2 = my$ and select a precision constant $C = 10^{80}$. Applying Lemma 2.1 to the lattice structure yields the computational estimate $|\Gamma_2| > 4.12 \times 10^{-45}$. Comparing this result directly with the geometric bound in (3.10) isolates the index as $m \leq 530$.

Conversely, if $m \in [0, 782]$ and $m \neq 0$, setting the operational lattice parameters back to $X = (1.94 \times 10^{18})^2$ and deploying a scaling size $C = 10^{180}$ allows us to evaluate Γ_1. Invoking Lemma 2.1 gives the verified lower

bound $|\Gamma_1| > 9.35 \times 10^{-140}$. Thus, matching this output back with inequality (3.10) yields $n \leq 1340$. From both analytical trajectories, we can conclude that the unified maximum system boundary safely reduces to:

$$L \leq 1340.$$

We now repeat this reduction process with this significantly lowered absolute bound of L to obtain the final optimized threshold for l. Under the localized assumption that $l > 50$, the primary inequalities in (3.8) and (3.10) reduce to the final numeric quotients:

$$|\Gamma_3| < \frac{42150}{(1.32)^l} \tag{3.11}$$

and

$$|\Gamma_i| < \frac{4215}{(1.32)^{n_i}}, \qquad i = 1 \text{ or } 2, \tag{3.12}$$

where $n_1 = n$ and $n_2 = m$. Finally, executing the LLL algorithm for the third time on the transitional form Γ_3 with the newly downscaled bounds of X and C provides the definitive lower bound $|\Gamma_3| > 2.89 \times 10^{-9}$. Comparing this analytical output against the exponential decay in inequality (3.11), we have the absolute minimized bound:

$$l < 112.$$

As we have rigorously established the optimized bound $l < 112$, two primary tactical sub-cases emerge to isolate the global limits. For the first structural partition where $n \in [3, 112]$, applying Lemma 2.1 to the companion linear form Γ_2 in our computer algebra environment yields the explicit computational estimate $|\Gamma_2| > 5.14 \times 10^{-10}$. Conversely, evaluating the alternative partition where $m \in [0, 112]$ on the lattice framework gives the lower verification bound $|\Gamma_1| > 1.22 \times 10^{-21}$. Comparing both analytical algorithmic outputs directly with the upper exponential constraints established in inequality (3.12), we safely deduce that the global parameter is tightly bounded by:

$$L \leq 210.$$

Consequently, by utilizing the explicit algebraic growth properties under the structural domain $L \leq 210$ along with the asymptotic threshold condition, we evaluate the maximum possible magnitude of the exponents. This yields the functional inequalities:

$$2^y \leq a_N^y \alpha_N^{my} = P_n^x \leq P_{210} \leq \alpha^{209} < (1.33)^{209},$$

which immediately gives rise to the secure bound $y \leq 85$. Following a completely symmetric analytical argument for the counter-component, we obtain the coordinate bound $x \leq 116$.

To find all the explicit structural solutions of our main Diophantine equation (1.1), we now restrict the search spaces for the index variables n, m and the exponents x, y to the following finite ranges:

$$n \in [3, 210], \quad m \in [4, 210] \cup \{0\}, \quad x \in [1, 116], \quad y \in [1, 85].$$

An exhaustive grid search executed across the restricted localized boundaries for the Padovan–Narayana Diophantine equation $(P_n)^x = (N_m)^y$ reveals that the nontrivial complete set of integer solution tuples (n, m, x, y) consists of the following combinations:

$$(n, m) \in \{(3,4), (5,5), (6,6), (9,8)\} \text{ with } x = y, \quad \{(3,6), (4,6), (5,8)\} \text{ with } x = 2y,$$

$$\{(6,4), (9,5), (11,6)\} \text{ with } y = 2x \quad \text{and} \quad \{(11,4)\} \text{ with } y = 4x.$$

This completes the proof.

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