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Some New Hardy Type Inequalities for Different Classes of Convex Functions

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Abstract. Hardy-type inequalities play an important role in mathematical analysis and have been extensively studied due to their wide range of applications in inequality theory and related fields. In recent years, various convexity generalizations have been used as effective tools to obtain new integral inequalities and extend classical results. In this paper, we have obtained some new Hardy-type inequalities that generalize certain known Hardy-type inequalities in the literature by utilizing different convexity classes, such as $s-$convex, quasi$-$convex, and $\eta-$convex functions, respectively. Also, our results reduce to previous work on classical convexity for specific choices of the parameters in the definitions of $s-$convexity and $\eta-$convexity.

1. Introduction

Hardy's classical integral inequality, first formulated in the early 20th century, is given as follows [6]:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x \hbar(s)\,ds\right)^p dx \le q^p \int_0^\infty \hbar^p(x)\,dx,$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x > 0$, f is a nonnegative integrable function on $(0, \infty)$. The constant $q = p(p-1)^{-1}$ is the best possible.

In recent years, Hardy-type inequalities have continued to attract considerable attention due to their broad applicability in mathematical analysis, integral inequalities, and fractional calculus. Various extensions have been established by employing different generalized convexity classes, including preinvex, harmonically convex, exponentially convex, and other generalized convex functions. At the same time, fractional integral operators have provided an effective framework for deriving refined Hardy-type inequalities with applications to fractional differential equations, numerical analysis, and mathematical physics. These developments demonstrate that generalized convexity and fractional calculus constitute two active research directions in the modern theory of Hardy-type inequalities. For Hardy integral inequalities which lead to many papers dealing with the alternative proofs, various generalizations, and numerous variants and applications in analysis see, [1–3, 5, 7, 8, 11–14, 16, 17].

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Building on the foundational results of Hardy-type inequalities, convexity theory offers a powerful and versatile framework for investigating a wide range of problems in both pure and applied mathematics. Over the years, numerous generalizations of convex sets and convex functions have been introduced, providing essential tools for analyzing complex structures and functional relationships. Convexity has played a central role in the development of inequalities, enabling sharper estimates and broader generalizations. Its applications extend beyond classical analysis to areas such as fractional and quantum calculus, stochastic processes, time-scale calculus, and fractal analysis, highlighting the deep connections between convexity and modern mathematical research.

There are many convexity classes in the literature. Now, let's continue by giving the definitions of $s-$convex function, quasi$-$convex function and $\eta-$convex function, which we will use in this article, respectively.

Definition 1.1. [4] *A function $\hbar : \mathbb{R}^+ \to \mathbb{R}$, is said to be $s-$convex in the second sense class as follows*

$$\hbar(\tau\theta + (1-\tau)\vartheta) \leq \tau^s \hbar(\theta) + (1-\tau)^s \hbar(\vartheta)$$

for all $\tau \in [0,1]$, $\theta, \vartheta \geq 0$ and for some fixed $s \in (0,1]$.

Definition 1.2. [10] *Let $\hbar : \imath \subseteq \mathbb{R} \to \mathbb{R}$ for all $\tau \in [0,1]$ and $\theta, \vartheta \in I$, if the following inequality*

$$\hbar(\tau\theta + (1-\tau)\vartheta) \leq \max\{\hbar(\theta), \hbar(\vartheta)\}$$

holds, then $\hbar$ is called a quasi$-$convex function on I.

Definition 1.3. [9] *The function $\hbar : [r,s] \to \mathbb{R}$ is said to be $\eta-$convex (or convex with respect to η) if the inequality*

$$\hbar(\tau\theta + (1-\tau)\vartheta) \leq \hbar(\vartheta) + \tau\eta(\hbar(\theta), \hbar(\vartheta))$$

holds for all $\theta, \vartheta \in [r,s]$, $\tau \in [0,1]$, and η is defined by $\eta : \hbar([r,s]) \times \hbar([r,s]) \to \mathbb{R}$.

Given that $A > B \geq 0$ and $p > 1$, the following elementary inequalities, which are a consequence of the convexity of the function $\hbar(x) = x^p$, are well known in the literature:

$$(A-B)^p \leq A^p - B^p, \tag{1}$$

and

$$(A+B)^p \leq 2^{p-1}(A^p + B^p). \tag{2}$$

The main contribution of this paper is twofold. First, we establish several new Hardy-type integral inequalities for the classes of s-convex, quasi-convex, and η-convex functions, which considerably extend the classical Hardy-type inequalities available in the literature. Second, unlike the results in [14], which are restricted to ordinary convex functions, the present work develops independent estimates for each generalized convexity class by directly exploiting their structural properties. Consequently, the obtained inequalities are not immediate consequences of the classical convex case but constitute genuine extensions that recover previously known results only for particular parameter choices.

2. Main Results

2.1. Hardy type inequalities for s−convex functions

Theorem 2.1. *Let $\hbar : [\theta, \vartheta] \subset [0,\infty) \to \mathbb{R}$ be a nonnegative $s-$convex function in the second sense for some fixed on $s \in (0,1]$ and $p > 1$. If $\omega : [\theta,\vartheta] \to [0,\infty)$ is an integrable function, then the following inequalities hold:*

$$\int_\theta^\vartheta \omega(\tau) T_1^p \hbar(\tau)\, d\tau \leq \frac{1}{(s+1)^p} \int_\theta^\vartheta \omega(\tau)(\tau-\theta)^p (\hbar(\theta) + \hbar(\tau))^p\, d\tau \tag{3}$$

and

$$\int_{\theta}^{\vartheta} \omega(\tau) T_2^p \hbar(\tau)\, d\tau \leq \frac{1}{(s+1)^p} \int_{\theta}^{\vartheta} \omega(\tau)(\vartheta-\tau)^p (\hbar(\vartheta)+\hbar(\tau))^p\, d\tau \tag{4}$$

where

$$T_1\hbar(\tau) = \int_{\theta}^{\tau} \hbar(k)\, dk \quad \textit{and} \quad T_2\hbar(\tau) = \int_{\tau}^{\vartheta} \hbar(k)\, dk.$$

Proof. By virtue of the $s-$convexity of $\hbar$ in the second sense on the subintervals $[\theta, \tau]$ and $[\tau, \vartheta]$ of $[\theta, \vartheta]$, one can write

$$\begin{aligned} T_1\hbar(\tau) = \int_{\theta}^{\tau} \hbar(k)\, dk &= \int_{\theta}^{\tau} \hbar\left(\frac{\tau-k}{\tau-\theta}\theta + \frac{k-\theta}{\tau-\theta}\tau\right) dk \\ &\leq \int_{\theta}^{\tau} \left(\left(\frac{\tau-k}{\tau-\theta}\right)^s \hbar(\theta) + \left(\frac{k-\theta}{\tau-\theta}\right)^s \hbar(\tau)\right) dk \\ &= \frac{\tau-\theta}{s+1}(\hbar(\theta)+\hbar(\tau)) \end{aligned} \tag{5}$$

and

$$\begin{aligned} T_2\hbar(t) = \int_{\tau}^{\vartheta} \hbar(k)\, dk &= \int_{\tau}^{\vartheta} \hbar\left(\frac{\vartheta-k}{\vartheta-\tau}\tau + \frac{k-\tau}{\vartheta-\tau}\vartheta\right) dk \\ &\leq \int_{\tau}^{\vartheta} \left(\left(\frac{\vartheta-k}{\vartheta-\tau}\right)^s \hbar(\tau) + \left(\frac{k-\tau}{\vartheta-\tau}\right)^s \hbar(\vartheta)\right) dk \\ &= \frac{\vartheta-\tau}{s+1}(\hbar(\vartheta)+\hbar(\tau)). \end{aligned} \tag{6}$$

Utilizing (5) and (6), it follows that

$$\int_{\theta}^{\vartheta} \omega(\tau) T_1^p \hbar(\tau)\, d\tau \leq \frac{1}{(s+1)^p} \int_{\theta}^{\vartheta} \omega(\tau)(\tau-\theta)^p (\hbar(\theta)+\hbar(\tau))^p\, d\tau$$

and

$$\int_{\theta}^{\vartheta} \omega(\tau) T_2^p \hbar(\tau)\, d\tau \leq \frac{1}{(s+1)^p} \int_{\theta}^{\vartheta} \omega(\tau)(\vartheta-\tau)^p (\hbar(\vartheta)+\hbar(\tau))^p\, d\tau.$$

So the desired inequalities is obtained. □

Remark 2.2. *If we choose $s = 1$ in Theorem 2.1, then we obtain [15, Theorem 2.1]. This shows that our result is a genuine extension of the classical Hardy-type inequality for convex functions, since s-convexity coincides with ordinary convexity when $s = 1$.*

Theorem 2.3. *Under the same assumptions in Theorem 2.1, the following inequality holds:*

$$\begin{aligned} &\int_{\theta}^{\vartheta} \omega(\tau)\left(T_1^p \hbar(\tau) + T_2^p \hbar(\tau)\right) d\tau \\ &\leq \frac{2^{p-1}(\vartheta^p - \theta^p)}{(s+1)^p}[\hbar^p(\theta) + \hbar^p(\vartheta)] \int_{\theta}^{\vartheta} \omega(\tau)\, d\tau + \frac{2^p(\vartheta^p - \theta^p)}{(s+1)^p} \int_{\theta}^{\vartheta} \omega(\tau)\hbar^p(\tau)\, d\tau. \end{aligned}$$

Proof. By summing inequalities (3) and (4) side by side and taking inequalities (1) and (2) into account, we obtain

$$\begin{aligned}
&\int_{\theta}^{\vartheta} \omega(\tau)\left[T_1^p\hbar(\tau) + T_2^p\hbar(\tau)\right] d\tau \\
&\le \frac{1}{(s+1)^p}\int_{\theta}^{\vartheta} \omega(\tau)[(\tau-\theta)^p + (\vartheta-\tau)^p][(\hbar(\theta)+\hbar(\tau))^p + (\hbar(\vartheta)+\hbar(\tau))^p]\, d\tau \\
&\le \frac{2^{p-1}(\vartheta^p-\theta^p)}{(s+1)^p}\int_{\theta}^{\vartheta} \omega(\tau)[\hbar^p(\theta)+\hbar^p(\vartheta)+2\hbar^p(\tau)]\, d\tau \\
&= \frac{2^{p-1}(\vartheta^p-\theta^p)}{(s+1)^p}[\hbar^p(\theta)+\hbar^p(\vartheta)]\int_{\theta}^{\vartheta} \omega(\tau)\, d\tau + \frac{2^{p}(\vartheta^p-\theta^p)}{(s+1)^p}\int_{\theta}^{\vartheta} \omega(\tau)\hbar^p(\tau)\, d\tau.
\end{aligned}$$

This completes the proof. □

Remark 2.4. *If we choose $s = 1$ in Theorem 2.3, then we obtain [15, Theorem 2.3]. Therefore, the present theorem generalizes the corresponding Hardy-type inequality from the classical convex setting to the broader class of s-convex functions.*

Theorem 2.5. *Under the hypothesis of Theorem 2.1, the following inequality holds:*

$$\begin{aligned}
&\int_{\theta}^{\vartheta} \omega(\tau)T_1^p\hbar(\tau)\, d\tau \qquad (7)\\
&\le \frac{p\hbar(\theta)}{(\vartheta-\theta)^s(s+1)^{p-1}}\int_{\theta}^{\vartheta} W(t)(\tau-\theta)^{p-1}(\hbar(\theta)+\hbar(\tau))^{p-1}[(\tau-\theta)^s + (\vartheta-\tau)^s]\, d\tau \\
&\quad + \frac{p}{(\vartheta-\theta)^s(s+1)^{p-1}}[\hbar(\vartheta)-\hbar(\theta)]\int_{\theta}^{\vartheta} W(\tau)(\tau-\theta)^{s+p-1}(\hbar(\theta)+\hbar(\tau))^{p-1}\, d\tau
\end{aligned}$$

where

$$W(\tau) = \int_{\tau}^{\vartheta} \omega(k)\, dk.$$

Proof. From the equality

$$\int_{\theta}^{\vartheta} \omega(\tau)T_1^p\hbar(\tau)\, d\tau = p\int_{\theta}^{\vartheta} W(\tau)[T_1^{p-1}\hbar(\tau)]\hbar(\tau)\, d\tau$$

obtained via integration by parts, and by virtue of (5) and $s-$convexity of $\hbar$ in the second sense, we get

$$
\begin{aligned}
&\int_{\theta}^{\vartheta} \omega(\tau) T_1^p \hbar(\tau)\, d\tau \\
&\leq \frac{p}{(s+1)^{p-1}} \int_{\theta}^{\vartheta} W(\tau)(\tau-\theta)^{p-1}(\hbar(\theta)+\hbar(\tau))^{p-1}\left(\left(\frac{\vartheta-\tau}{\vartheta-\theta}\right)^s \hbar(\theta) + \left(\frac{\tau-\theta}{\vartheta-\theta}\right)^s \hbar(\vartheta)\right) d\tau \\
&= \frac{p\hbar(\theta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} W(\tau)(\tau-\theta)^{p-1}(\vartheta-\tau)^s(\hbar(\theta)+\hbar(\tau))^{p-1}\, d\tau \\
&\quad + \frac{p\hbar(\vartheta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} W(\tau)(\tau-\theta)^{s+p-1}(\hbar(\theta)+\hbar(\tau))^{p-1}\, d\tau \\
&= \frac{p\hbar(\theta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} W(\tau)(\tau-\theta)^{p-1}(\hbar(\theta)+\hbar(\tau))^{p-1}[(\tau-\theta)^s + (\vartheta-\tau)^s]\, d\tau \\
&\quad + \frac{p}{(\vartheta-\theta)^s(s+1)^{p-1}} [\hbar(\vartheta)-\hbar(\theta)] \int_{\theta}^{\vartheta} W(\tau)(\tau-\theta)^{s+p-1}(\hbar(\theta)+\hbar(\tau))^{p-1}\, d\tau.
\end{aligned}
$$

So the proof of is completed. □

Remark 2.6. *If we choose $s = 1$ in Theorem 2.1, then we obtain [15, Theorem 2.5].*

Theorem 2.7. *Under the hypothesis of Theorem 2.1, the following inequality holds:*

$$
\begin{aligned}
&\int_{\theta}^{\vartheta} \omega(\tau) T_2^p \hbar(\tau)\, d\tau \\
&\leq \frac{p\hbar(\vartheta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{p-1}(\hbar(\vartheta)+\hbar(\tau))^{p-1}[(\vartheta-\tau)^s + (\tau-\theta)^s]\, d\tau \\
&\quad + \frac{p}{(\vartheta-\theta)^s(s+1)^{p-1}} [\hbar(\theta)-\hbar(\vartheta)] \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{s+p-1}(\hbar(\vartheta)+\hbar(\tau))^{p-1}\, d\tau
\end{aligned} \tag{8}
$$

where

$$H(\tau) = \int_{\theta}^{\tau} \omega(s)\, ds.$$

Proof. From the equality

$$\int_{\theta}^{\vartheta} \omega(\tau) T_2^p \hbar(\tau)\, d\tau = p \int_{\theta}^{\vartheta} H(\tau)[T_2^{p-1}\hbar(\tau)]\hbar(\tau)\, d\tau$$

obtained via integration by parts, and by virtue of (6) and $s-$convexity of $\hbar$ in the second sense, we have

$$
\begin{aligned}
&\int_{\theta}^{\vartheta} \omega(\tau) T_2^p \hbar(\tau)\, d\tau \\
&\leq \frac{p}{2^{p-1}} \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{p-1}(\hbar(\vartheta)+\hbar(\tau))^{p-1}\left(\left(\frac{\vartheta-\tau}{\vartheta-\theta}\right)^s \hbar(\theta)+\left(\frac{\tau-\theta}{\vartheta-\theta}\right)^s \hbar(\vartheta)\right) d\tau \\
&= \frac{p\hbar(\theta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{s+p-1}(\hbar(\vartheta)+\hbar(\tau))^{p-1}\, d\tau \\
&\quad + \frac{p\hbar(\vartheta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{p-1}(\tau-\theta)^s(\hbar(\vartheta)+\hbar(\tau))^{p-1}\, d\tau \\
&= \frac{p\hbar(\vartheta)}{(\vartheta-\theta)^s(s+1)^{p-1}} \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{p-1}(\hbar(\vartheta)+\hbar(\tau))^{p-1}[(\vartheta-\tau)^s+(\tau-\theta)^s]\, d\tau \\
&\quad + \frac{p}{(\vartheta-\theta)^s(s+1)^{p-1}}[\hbar(\theta)-\hbar(\vartheta)] \int_{\theta}^{\vartheta} H(\tau)(\vartheta-\tau)^{s+p-1}(\hbar(\vartheta)+\hbar(\tau))^{p-1}\, d\tau.
\end{aligned}
$$

This completes the proof. □

Remark 2.8. *If we choose $s = 1$ in Theorem 2.1, then we obtain [15, Theorem 2.6].*

2.2. *Hardy type inequalities for quasi-convex functions*

Theorem 2.9. *Let $\hbar : [\theta, \vartheta] \subset [0, \infty) \to \mathbb{R}$ be a nonnegative quasi−convex function and $p > 1$. If $\omega : [\theta, \vartheta] \to [0, \infty)$ is an integrable function, then the following inequalities hold:*

$$\int_{\theta}^{\vartheta} \omega(\tau) T_1^p \hbar(\tau)\, d\tau \leq \int_{\theta}^{\vartheta} \omega(\tau)(\tau-\theta)^p \max\{\hbar(\theta), \hbar(\tau)\}^p\, d\tau \tag{9}$$

and

$$\int_{\theta}^{\vartheta} \omega(\tau) T_2^p \hbar(\tau)\, d\tau \leq \int_{\theta}^{\vartheta} \omega(\tau)(\vartheta-\tau)^p \max\{\hbar(\tau), \hbar(\vartheta)\}^p\, d\tau \tag{10}$$

where

$$T_1\hbar(\tau) = \int_{\theta}^{\tau} \hbar(k)\, dk \quad \textit{and} \quad T_2\hbar(\tau) = \int_{\tau}^{\vartheta} \hbar(k)\, dk.$$

Proof. By virtue of the quasi−convexity of $\hbar$ on the subintervals $[\theta, \tau]$ and $[\tau, \vartheta]$ of $[\theta, \vartheta]$, one can write

$$
\begin{aligned}
T_1\hbar(\tau) = \int_{\theta}^{\tau} \hbar(k)\, dk &= \int_{\theta}^{\tau} \hbar\left(\frac{\tau-k}{\tau-\theta}\theta + \frac{k-\theta}{\tau-\theta}\tau\right) dk \\
&\leq \int_{\theta}^{\tau} \max\{\hbar(\theta), \hbar(\tau)\} dk \\
&= (\tau-\theta)\max\{\hbar(\theta), \hbar(\tau)\}
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
T_2\hbar(\tau) = \int_{\tau}^{\vartheta} \hbar(k)\, dk &= \int_{\tau}^{\vartheta} \hbar\left(\frac{\vartheta-k}{\vartheta-\tau}\tau + \frac{k-\tau}{\vartheta-\tau}\vartheta\right) dk \\
&\leq \int_{\tau}^{\vartheta} \max\{\hbar(\tau), \hbar(\vartheta)\} dk \\
&= (\vartheta-\tau)\max\{\hbar(\tau), \hbar(\vartheta)\}.
\end{aligned} \tag{12}
$$

Utilizing (11) and (12), it follows that

$$\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \leq \int_\theta^\vartheta \omega(\tau)(\tau-\theta)^p \max\{\hbar(\theta),\hbar(\tau)\}^p\,d\tau$$

and

$$\int_\theta^\vartheta \omega(\tau)T_2^p\hbar(\tau,d\tau \leq \int_\theta^\vartheta \omega(\tau)(\vartheta-\tau)^p \max\{\hbar(\tau),\hbar(\vartheta)\}^p\,d\tau.$$

So the desired inequalities is obtained. □

Theorem 2.10. *Under the same assumptions in Theorem 2.9, the following inequality holds:*

$$\begin{aligned}&\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \\ &\leq p\max\{\hbar(\theta),\hbar(\vartheta)\}\int_t heta^\vartheta W(\tau)(\tau-\theta)^{p-1}\max\{\hbar(\theta),\hbar(\tau)\}^{p-1}d\tau\end{aligned} \tag{13}$$

where

$$W(\tau)=\int_\tau^\vartheta \omega(k)\,dk.$$

Proof. From the equality

$$\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau = p\int_\theta^\vartheta W(\tau)[T_1^{p-1}\hbar(\tau)]\hbar(\tau)\,d\tau$$

obtained via integration by parts, and by virtue of (11) and quasi−convexity of $\hbar$ on $[\theta,\vartheta]$, we have

$$\begin{aligned}&\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \\ &\leq p\int_\theta^\vartheta W(\tau)(\tau-\theta)^{p-1}\max\{\hbar(\theta),\hbar(\tau)\}^{p-1}\max\{\hbar(\theta),\hbar(\vartheta)\}d\tau.\end{aligned}$$

This completes the proof of the inequality (13). □

Theorem 2.11. *Under the same assumptions in Theorem 2.9, the following inequality holds:*

$$\begin{aligned}&\int_\theta^\vartheta \omega(\tau)T_2^p\hbar(\tau)\,d\tau \\ &\leq p\max\{\hbar(\theta),\hbar(\vartheta)\}\int_\theta^\vartheta H(\tau)(\vartheta-\tau)^{p-1}\max\{\hbar(\tau),\hbar(\vartheta)\}^{p-1}\,d\tau\end{aligned} \tag{14}$$

where

$$H(\tau)=\int_\theta^\tau \omega(s)\,ds.$$

Proof. From the equality

$$\int_\theta^\vartheta \omega(\tau)T_2^p\hbar(\tau)\,d\tau = p\int_\theta^\vartheta H(\tau)[T_2^{p-1}\hbar(\tau)]\hbar(\tau)\,d\tau$$

obtained via integration by parts, and by virtue of (12) and quasi−convexity of $\hbar$ on $[\theta,\vartheta]$, we have

$$\begin{aligned}&\int_\theta^\vartheta \omega(\tau)T_2^p\hbar(\tau)\,d\tau\\ &\le p\int_\theta^\vartheta H(\tau)(\vartheta-\tau)^{p-1}\max\{\hbar(\tau),\hbar(\vartheta)\}^{p-1}\max\{\hbar(\theta),\hbar(\vartheta)\}d\tau.\end{aligned}$$

This completes the proof. □

Remark 2.12. *This result extends the classical Hardy-type inequality to the quasi-convex setting, thereby enlarging the class of admissible functions while preserving the structure of the Hardy inequality.*

2.3. Hardy type inequalities for η-convex functions

Theorem 2.13. *Let* $\hbar:[\theta,\vartheta]\subset[0,\infty)\to\mathbb{R}$ *be a nonnegative* η−*convex function and* $p>1$. *If* $\omega:[\theta,\vartheta]\to[0,\infty)$ *is an integrable function, then the following inequalities hold:*

$$\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \le \int_\theta^\vartheta \omega(\tau)\left((\tau-\theta)\hbar(\tau)+\frac{\tau-\theta}{2}\eta(\hbar(\theta),\hbar(\tau))\right)^p d\tau \tag{15}$$

and

$$\int_\theta^\vartheta \omega(\tau)T_2^p\hbar(\tau)\,d\tau \le \int_\theta^\vartheta \omega(\tau)\left((\vartheta-\tau)\hbar(\vartheta)+\frac{\vartheta-\tau}{2}\eta(\hbar(\tau),\hbar(\vartheta))\right)^p d\tau \tag{16}$$

where

$$T_1\hbar(\tau)=\int_\theta^\tau \hbar(k)\,dk \quad \textit{and} \quad T_2\hbar(\tau)=\int_\tau^\vartheta \hbar(k)\,dk.$$

Proof. By virtue of the η−convexity of $\hbar$ on the subintervals $[\theta,\tau]$ and $[\tau,\vartheta]$ of $[\theta,\vartheta]$, one can write

$$\begin{aligned}T_1\hbar(\tau)=\int_\theta^\tau \hbar(k)\,dk &= \int_\theta^\tau \hbar\left(\frac{\tau-k}{\tau-\theta}\theta+\frac{k-\theta}{\tau-\theta}\tau\right)dk\\ &\le \int_\theta^\tau\left(\hbar(\tau)+\left(\frac{\tau-k}{\tau-\theta}\right)\eta(\hbar(\theta),\hbar(\tau))\right)dk\\ &= (\tau-\theta)\hbar(\theta)+\frac{\tau-\theta}{2}\eta(\hbar(\theta),\hbar(\tau))\end{aligned} \tag{17}$$

and

$$\begin{aligned}T_2\hbar(\tau)=\int_\tau^\vartheta \hbar(k)\,dk &= \int_\tau^\vartheta \hbar\left(\frac{\vartheta-k}{\vartheta-\tau}\tau+\frac{k-\tau}{\vartheta-\tau}\vartheta\right)dk\\ &\le \int_\tau^\vartheta\left(\hbar(\tau)+\left(\frac{\vartheta-k}{\vartheta-\tau}\right)\eta(\hbar(\tau),\hbar(\vartheta))\right)dk\\ &= (\vartheta-\tau)\hbar(\vartheta)+\frac{\vartheta-\tau}{2}\eta(\hbar(\tau),\hbar(\vartheta)).\end{aligned} \tag{18}$$

Utilizing (17) and (18), it follows that

$$\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \le \int_\theta^\vartheta \omega(\tau)\left((\tau-\theta)\hbar(\tau)+\frac{\tau-\theta}{2}\eta(\hbar(\theta),\hbar(\tau))\right)^p d\tau$$

and

$$\int_\theta^\vartheta \omega(\tau)T_2^p\hbar(\tau)\,d\tau \le \int_\theta^\vartheta \omega(\tau)\left((\vartheta-\tau)\hbar(\vartheta)+\frac{\vartheta-\tau}{2}\eta(\hbar(\tau),\hbar(\vartheta))\right)^p d\tau$$

So the desired inequalities is obtained. □

Remark 2.14. *If we choose $\eta(\hbar(x),\hbar(y)) = \hbar(x)-\hbar(y)$ in Theorem 2.13, then we obtain [15, Theorem 2.1].*

Theorem 2.15. *Under the hypothesis of Theorem 2.13, the following inequality holds:*

$$\begin{aligned}
&\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \\
\le\ &\frac{p\hbar(\tau)}{2^{p-1}}\int_\theta^\vartheta W(\tau)(\tau-\vartheta)^{p-1}\left(2\hbar(\theta)+\eta(\hbar(\tau),\hbar(\theta))\right)^{p-1} d\tau \\
&+\frac{p}{(b-a)2^{p-1}}\eta(\hbar(\vartheta),\hbar(\theta))\int_\theta^\vartheta W(\tau)(\tau-\theta)^p\left(2\hbar(\theta)+\eta(\hbar(\tau),\hbar(\theta))\right)^{p-1} d\tau
\end{aligned} \tag{19}$$

where

$$W(\tau)=\int_\tau^\vartheta \omega(k)\,dk.$$

Proof. From the equality

$$\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau = p\int_a^b W(\tau)[T_1^{p-1}\hbar(\tau)]\hbar(\tau)\,d\tau$$

obtanied via integration by parts, and by virtue of (17) and η−convexity of $\hbar$ on $[\theta,\vartheta]$, we have

$$\begin{aligned}
&\int_\theta^\vartheta \omega(\tau)T_1^p\hbar(\tau)\,d\tau \\
\le\ & p\int_\theta^\vartheta W(\tau)(\tau-\theta)^{p-1}\left(\hbar(\theta)+\frac{1}{2}\eta(\hbar(\theta),\hbar(\tau))\right)^{p-1}\left(\hbar(\theta)+\frac{\tau-\theta}{\vartheta-\theta}\eta(\hbar(\vartheta),\hbar(\theta))\right)d\tau \\
=\ &\frac{pf(a)}{2^{p-1}}\int_\theta^\vartheta W(\tau)(\tau-\theta)^{p-1}\left(2\hbar(\theta)+\eta(\hbar(\tau),\hbar(\theta))\right)^{p-1} d\tau \\
&+\frac{p}{(b-a)2^{p-1}}\eta(\hbar(\vartheta),\hbar(\theta))\int_\theta^\vartheta W(\tau)(\tau-\theta)^p\left(2\hbar(\theta)+\eta(\hbar(\theta),\hbar(\tau))\right)^{p-1} d\tau.
\end{aligned}$$

So the proof is completed. □

Remark 2.16. *If we choose $\eta(\hbar(x),\hbar(y)) = \hbar(x)-\hbar(y)$ in Theorem 2.15, then we obtain [15, Theorem 2.5]. Hence, Theorem 2.8 recovers Theorem 2.1 of [14], confirming that the obtained inequality is a true extension of the classical result.*

Theorem 2.17. *Under the hypothesis of Theorem 2.13, the following inequality holds:*

$$\begin{aligned}
&\int_\theta^\vartheta \omega(\tau) T_2^p \hbar(\tau)\, d\tau \\
&\leq \frac{pf(b)}{2^{p-1}} \int_\theta^\vartheta H(\tau)(\vartheta-\tau)^{p-1} \left(2\hbar(\vartheta) + \eta(\hbar(\tau), \hbar(\vartheta))\right)^{p-1} d\tau \\
&\quad + \frac{p}{(\vartheta-\theta)2^{p-1}} \eta(\hbar(\theta), \hbar(\vartheta)) \int_\theta^\vartheta H(\tau)(\vartheta-\tau)^p \left(2\hbar(\vartheta) + \eta(\hbar(\tau), \hbar(\vartheta))\right)^{p-1} d\tau
\end{aligned} \tag{20}$$

where

$$H(\tau) = \int_\theta^\tau \omega(s)\, ds.$$

Proof. From the equality

$$\int_\theta^\vartheta \omega(\tau) T_2^p \hbar(\tau)\, d\tau = p \int_\theta^\vartheta H(\tau)[T_2^{p-1}\hbar(\tau)]\hbar(\tau)\, d\tau$$

obtained via integration by parts, and by virtue of (18) and $\eta-$convexity of $\hbar$ on $[\theta, \vartheta]$, we get

$$\begin{aligned}
&\int_\theta^\vartheta \omega(\tau) T_2^p \hbar(\tau)\, d\tau \\
&\leq p \int_\theta^\vartheta H(\tau)(\vartheta-\tau)^{p-1} \left(\hbar(\vartheta) + \frac{1}{2}\eta(\hbar(\tau), \hbar(\vartheta))\right)^{p-1} \left(\hbar(\vartheta) + \frac{\vartheta-\tau}{\vartheta-\theta}\eta(\hbar(\theta), \hbar(\vartheta))\right) d\tau \\
&= \frac{pf(b)}{2^{p-1}} \int_\theta^\vartheta H(\tau)(\vartheta-\tau)^{p-1} \left(2\hbar(\vartheta) + \eta(\hbar(\tau), \hbar(\vartheta)\right)^{p-1} d\tau \\
&\quad + \frac{p}{(\vartheta-\theta)2^{p-1}} \eta(\hbar(\theta), \hbar(\vartheta)) \int_\theta^\vartheta H(\tau)(\vartheta-\tau)^p \left(2\hbar(\vartheta) + \eta(\hbar(\tau), \hbar(\vartheta))\right)^{p-1} d\tau.
\end{aligned}$$

This completes the proof. □

Remark 2.18. *If we choose $\eta(\hbar(x), \hbar(y)) = \hbar(x) - \hbar(y)$ in Theorem 2.17, then we obtain [15, Theorem 2.6].*

3. Conclusion

In this study, we established several new Hardy-type integral inequalities by employing three important generalized convexity classes, namely s-convex, quasi-convex, and η-convex functions. The obtained inequalities extend a number of previously known Hardy-type inequalities for classical convex functions and demonstrate that the proposed approach provides a unified framework capable of treating different convexity structures within a common analytical setting. In particular, the derived results reduce to several well-known inequalities available in the literature for suitable choices of the defining parameters, confirming the consistency and generality of the proposed approach.

The proofs are based on the intrinsic properties of the corresponding convexity classes together with classical integral techniques and suitable estimates, leading to new bounds that are not direct consequences of the existing convex case. Consequently, the present results enlarge the applicability of Hardy-type inequalities to broader families of functions while preserving the structure of the classical inequalities.

The proposed framework also provides a useful basis for further developments in inequality theory. In particular, similar techniques may be employed to derive new Hardy-type inequalities for other generalized convexity classes, including preinvex, harmonically convex, exponentially convex, and strongly convex functions. Furthermore, it would be of considerable interest to investigate analogous inequalities involving fractional integral operators, weighted integral operators, and other generalized operators arising in fractional calculus and related areas of mathematical analysis.

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