TURKISH JOURNAL OF SCIENCE
VOLUME 11, ISSUE 1, 1–11
ISSN: 2587–0971

Numerical Solution of Inverse Nodal Problems for Singular Diffusion Equation

Abdullah Ergün[a]

[a] *Sivas Vocational School of Technical Sciences, Sivas Cumhuriyet University, 58140, Sivas, Türkiye*

Abstract. This study focuses on the inverse nodal problem for a singular diffusion equation that involves eigenparameter-dependent terms and internal discontinuities. First, we derive reconstruction formulas for the potential function $q(x)$ and the parameter β using nodal data, applying the Chebyshev interpolation method in the limiting process. Next, we establish the stability of the inverse problem through rigorous analysis. Finally, approximate solutions are obtained using a numerical approach based on Chebyshev polynomials. To evaluate the effectiveness of the method, numerical experiments are conducted with MATLAB, comparing the results to those obtained through classical solution techniques.

1. Introduction

Many mathematical models, particularly in quantum and classical mechanics, are commonly formulated using Hamiltonian or Schrödinger operators with singular coefficients. These models often lead to the study of spectral inverse problems involving differential operators with generalized or singular coefficients. In fact, a large class of problems in mathematical physics can be reduced to such formulations.
A representative example is the problem of stationary vibrations in a spring-connected homogeneous string fixed at both ends, with density defined by $S'(x) = a\delta(x - x_0)$, where $\delta(x)$ denotes the Dirac delta function, and stiffness $S(x)$ is evaluated at point x_0. The domain of this operator is given by:

$$\begin{aligned} &D(L_0) = \Big\{ y(x) \in W_2^2[0,1] : \\ &y'(x_0+0) - y'(x_0-0) = ay(x_0),\ x_0 \in (0,1)\,;\ y(0) = 0 = y(1) \Big\} \end{aligned}$$

This system is represented by the differential operator $L_0 = -\frac{d^2}{dx^2}$ in the Hilbert space $L_2[0,1]$. There exists a comprehensive theory on the proper (regular) definition of such operators and the analysis of their spectral characteristics $[2-4]$.
In this study, we consider a class of Sturm–Liouville-type quadratic pencil differential equations given by:

$$ly := -y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 y, \quad x \in [0,\pi] / \{a\} \tag{1}$$

with boundary conditions:

$$y'(0) = hy(0) \tag{2}$$

Corresponding author: A. E.: aergun@cumhuriyet.edu.tr ORCID:0000-0002-2795-8097
Received: 18 January 2026; Accepted: 20 February 2026; Published: 30 April 2026
Keywords. Inverse nodal problem, Sturm-Liouville equation, stability, Chebyshev interpolation method, approximate solutions.
2010 *Mathematics Subject Classification*. 34A55, 34B24, 41A10
Cited this article as: Ergün, A. (2026). Numerical Solution of Inverse Nodal Problems for Singular Diffusion Equation. Turkish Journal of Science, 11(1), 1-11.

$$y'(\pi) = -Hy(\pi) \tag{3}$$

Here, $q(x) \in L_2[0,1]$ is a real-valued function,λ is a spectral parameter, and $p(x) = \beta\delta(x-a)$, where $h, H, \beta \neq 0$ and a are real constants.
Definition 1. A function $y(x) \in W_2^2(\Omega)\,(\Omega = [0,\pi]\,/\,\{a\})$, is called a solution of the equation if it satisfies:

$$-y'' + q(x)\,y = \lambda^2 y, \quad x \in [0,\pi]\,/\,\{a\} \tag{4}$$

along with the discontinuity condition

$$y'(a+0) - y'(a-0) = \lambda\beta y(a) \tag{5}$$

Moreover, if for all such nontrivial functions $y(x)$ satisfying the boundary and discontinuity conditions, the following inequality holds:

$$h\left|y(0)\right|^2 + H\left|y(\pi)\right|^2 + \int_0^\pi \left\{\left|y'(x)\right|^2 + q(x)\left|y(x)\right|^2\right\} dx > 0 \tag{6}$$

Sturm–Liouville operators with singular coefficients arise frequently in both theoretical and applied physics. In earlier studies[5 – 13], the spectral properties of such operators with separated or non-separated boundary conditions involving spectral parameters were explored, and uniqueness theorems for the corresponding inverse problems were established. Similar investigations were carried out for Schrödinger-type equations with singular coefficients under spectral boundary conditions [14 – 17].
One prominent approach to solving inverse problems in this context is through nodal data specifically, the zeros of eigenfunctions, known as nodal points. The process of recovering the coefficients of the differential operator from the asymptotic behavior of these nodal points defines the inverse nodal problem. McLaughlin[18] pioneered this area by demonstrating the uniqueness of solutions for Sturm–Liouville operators. Further computational developments were provided by Hald and McLaughlin [19] while Yang introduced a systematic algorithm for inverse nodal problems under separated boundary conditions[20]. Subsequent works extended these results of boundary conditions [21 – 32] .
The first treatment of inverse nodal problems involving discontinuous conditions in Sturm–Liouville systems was presented by Shieh and Yurko and these findings were later expanded by Yang who proved uniqueness and stability theorems for cases with interior discontinuities located at $a = \frac{1}{2}$.
Koyunbakan and Mosazadeh [34] addressed Sturm–Liouville problems with spectral boundary conditions and discontinuities at $a = \frac{\pi}{2}$ providing a constructive procedure for recovering the potential function using nodal lengths, and analyzing Lipschitz stability of the solution. Further studies extended these investigations to various types of regular Sturm–Liouville and Dirac operators[35 – 38].
However, existing results generally assume that the point of discontinuity is fixed and rational. In contrast, this paper considers discontinuities occurring at irrational locations of the form $a_r = \pi r, (r \in (0,1) \cap \mathrm{Q})$ and presents a uniqueness theorem for the inverse nodal problem in such cases. An explicit algorithm is also proposed for reconstructing the coefficients of the operator using asymptotic nodal data.

2. Inverse Nodal Problem

Theorem 1. ([40]) The asymptotic representations of the solutions of problem (1)-(3) that satisfy condition (4) are given as follows:
for$x < a$,

$$\varphi(x,\lambda) = \cos\lambda x + \left(h + \tfrac{1}{2}\int_0^x q(t)\,dt\right)\tfrac{\sin\lambda x}{\lambda} + o\left(\tfrac{e^{|\tau|x}}{|\lambda|}\right) \tag{7}$$

$$\varphi'(x,\lambda) = -\lambda\sin\lambda x + \left(h + \tfrac{1}{2}\int_0^x q(t)\,dt\right)\cos\lambda x + o\left(e^{|\tau|x}\right) \tag{8}$$

for$x > a$,

$$\begin{aligned}&\varphi(x,\lambda)=\\&\cos\lambda x+\tfrac{1}{2}\beta\left(\sin\lambda x-\sin\lambda\left(2a-x\right)\right)+\omega_0(x)\cos\lambda x+\omega_1(x)\tfrac{\cos\lambda x}{\lambda}\\&+\left(1-2\beta^2\right)\int_0^a q(t)\,dt\tfrac{\sin\lambda(2a-x)}{4\lambda}+\omega_2(x)\tfrac{\cos\lambda(2a-x)}{2\lambda}+o\left(\tfrac{e^{|\tau|x}}{|\lambda|}\right)\end{aligned}\tag{9}$$

$$\begin{aligned}&\varphi'(x,\lambda)=\\&\lambda\left[-\sin\lambda x+\tfrac{1}{2}\beta\left(\cos\lambda x+\cos\lambda\left(2a-x\right)\right)\right]+\omega_0(x)\cos\lambda x-\omega_1(x)\sin\lambda x\\&+\omega_2(x)\sin\lambda\left(2a-x\right)-\tfrac{1}{4}\left(1-2\beta^2\right)\int_0^a q(t)\,dt\cos\lambda\left(2a-x\right)+o\left(e^{|\tau|x}\right)\end{aligned}\tag{10}$$

where:

$$\omega_0(x)=h-\frac{1}{4}\beta\int_0^a q(t)\,dt+\frac{1}{2}\int_0^x q(t)\,dt,\omega_1(x)=-\frac{1}{2}\beta\left(h-\frac{1}{2}\int_0^a q(t)\,dt+\int_0^x q(t)\,dt\right)$$

$$\omega_2(x)=\frac{1}{2}\beta\left(h-\frac{3}{2}\int_0^a q(t)\,dt+\int_0^x q(t)\,dt\right),\omega_3(x)=\frac{1}{4}\left(1-\beta^2\right)\int_0^a q(t)\,dt$$

Let us define the auxiliary function:

$$\Delta_0(\lambda)=\lambda\left[-\sin\lambda\pi+\frac{1}{2}\beta\left(\cos\lambda\pi+\cos\lambda\left(2a-\pi\right)\right)\right]\tag{11}$$

Following the analytical approach presented by Ya in [40], it is established that the roots of the equation $\Delta_0(\lambda)$ admit the asymptotic representation:
$\lambda_n^0=n+h_n$, with $\sup\limits_n|hn|=h<\infty$
Applying the same method to the characteristic equation $\Delta(\lambda)=0$, and utilizing the expression obtained from $\Delta(\lambda)$, one derives the refined asymptotic approximation:

$$\lambda_n=\lambda_n^0+o(1)\tag{12}$$

in the sense of Rouche's Theorem.
Let us now define the contour $G_n:=\left\{\lambda\in C:|\lambda|=\left|\lambda_n^0\right|+\frac{\delta}{2}\right\}$
Moreover, according to [43], the difference between $\Delta(\lambda)$ and $\Delta_0(\lambda)$ satisfies the bound:

$$\Delta(\lambda)-\Delta_0(\lambda)=O\left(\exp|Im\lambda|\pi\right),\ |\lambda|\to\infty$$

Therefore, for sufficiently large n and $\lambda\in G_n$, one obtains:

$$|\Delta(\lambda)-\Delta_0(\lambda)|<\frac{1}{2C_\delta}\exp\left(|Im\lambda|\pi\right).$$

On the other hand, we also have:

$$|\Delta_0(\lambda)|\geq C_\delta|\lambda|\exp\left(|Im\lambda|\pi\right)>|\Delta(\lambda)-\Delta_0(\lambda)|$$

Hence, by Rouche's theorem, $\Delta(\lambda)$ and $\Delta_0(\lambda)$ must have the same number of zeros (counting multiplicity) within the contour G_n, and that number is exactly $n+1$ corresponding to the zeros $\lambda_0,\lambda_1,...,\lambda_n$.
Furthermore, by applying Rouche's theorem again in a neighborhood around each individual λ_n^0, namely the disk

$$C_\delta==\left\{\lambda:\left|\lambda-\lambda_n^0\right|\leq\delta\right\}$$

we conclude that, for sufficiently large n, each such disk contains precisely one zero λ_n of $\Delta(\lambda)$. Since δ is arbitrary but small, it follows that
$\lambda_n=\lambda_n^0+\varepsilon_n$ where $\varepsilon_n=o(1),\ n\to\infty$.

Lemma 1.([40])

For sufficiently large n and uniformly with respect to $j = j(n)$, the nodal points $x_n^{j(n)}$ satisfy the following asymptotic formulas:if $x_n^{j(n)} \in (0, a)$,

$$x_n^{j(n)} = \frac{\left(j - \frac{1}{2}\right)\pi}{\lambda_n^0} + \left\{k + \frac{1}{2}\int_0^{x_n^{j(n)}} q(t)\,dt - d_n x_n^{j(n)}\right\} \frac{1}{\left(\lambda_n^0\right)^2} + o\left(\frac{1}{\left(\lambda_n^0\right)^2}\right)$$

if $x_n^{j(n)} \in (a, \pi)$,

$$x_n^{j(n)} = \frac{\left(j - \frac{1}{2}\right)\pi}{\lambda_n^0} + \frac{1}{\lambda_n^0}\arctan\left(\frac{B_n}{A_n}\right) - \left\{\frac{B_n}{A_n^2} D_n\left(x_n^{j(n)}\right) + \frac{C_n\left(x_n^{j(n)}\right)}{A_n}\right\} \frac{1}{\left(\lambda_n^0\right)^2} + o\left(\frac{1}{\left(\lambda_n^0\right)^2}\right).$$

$$\frac{1}{\lambda_{n_k}^0} = \frac{1}{n_k} - \frac{h_{n_k}}{(n_k)^2} + o\left(\frac{1}{(n_k)^2}\right)$$

Definition 1
The set $X(L) := \left\{x_n^{j(n)}, n = 1, 2, ...; j(n) = 1, ..., n\right\}$is referred to as the *nodal set* corresponding to the boundary value problem defined by equations L.Let $X_0(L)$be a subset of $X(L)$. This subset is referred to as a *twin set* if, for each element $x_{n_k}^{j(n_k)} \in X_0(L)$, the set also includes at least one of its neighboring nodal points either $x_{n_k}^{j(n_k)-1}$or $x_{n_k}^{j(n_k)+1}$or both.
Now consider the set
$X_0(L) = \left\{x_{n_k}^{j(n_k)} : n_k = 1, 2, ...; j(n_k) = 1, 2..., n_k\right\}$which forms a subsequence of the nodal points $x_n^{j(n)}$that is dense in the interval $(0, \pi)$.In light of previous results, the existence of such a set is evident.

Lemma 2. Let $X(L)$and $X\left(\tilde{L}\right)$be two dense and twin subsets of nodal points. Suppose that $X(L) = X\left(\tilde{L}\right))$, and for each t, there exist indices n_k,$\tilde{n}_k$ and n_t,$\tilde{n}_t$ such that $X_{n_k}(L) = X_{\tilde{n}_k}\left(\tilde{L}\right)$. Then for all sufficiently large t, it follows that $n_k = \tilde{n}_k$.

Proof. Assume, for the sake of contradiction, that $n_k \neq \tilde{n}_k$Without loss of generality, suppose $n_t \geq \tilde{n}_t$. Since$X_{n_k}(L) = \tilde{X}_{n_k}\left(\tilde{L}\right)$there must exist indices $\tilde{z}_1$ and $\tilde{z}_2$such that
$x_{n_k}^{j(n_k)} = \tilde{x}_{\tilde{n}_k}^{\tilde{z}_1}$ and $x_{n_k}^{j(n_k)+1} = \tilde{x}_{\tilde{n}_k}^{\tilde{z}_2}$.

$$x_{n_k}^{j(n_k)+1} - x_{n_k}^{j(n_k)} = \tilde{x}_{\tilde{n}_k}^{\tilde{z}_2} - \tilde{x}_{\tilde{n}_k}^{\tilde{z}_1}$$

This yields the relation:

$$\frac{\tilde{n}_k}{n_k(\tilde{z}_2 - \tilde{z}_1)}\left(1 - \frac{h_{n_k}}{n_k}\right) = \frac{\tilde{n}_k - h_{\tilde{n}_{t,k}}}{\tilde{n}_k} + o\left(\frac{1}{\tilde{n}_k}\right)$$

As n_k, $\tilde{n}_k \to \infty$. The above implies that $\tilde{z}_2 = \tilde{z}_1 + 1$and consequently $n_k = \tilde{n}_k$
Therefore, it must be that
$x_{n_k}^{j(n_k)} = \tilde{x}_{\tilde{n}_k}^{\tilde{z}_1}$ and $x_{n_k}^{j(n_k)+1} = \tilde{x}_{\tilde{n}_k}^{\tilde{z}_2}$.

$$x_{n_k}^{j(n_k)} = \frac{\left(j(n_k) - \frac{1}{2}\right)\pi}{n_k}\left(1 - \frac{h_{n_k}}{n_k}\right) + o\left(\frac{1}{n_k}\right)$$

and

$$\tilde{x}_{n_k}^{\tilde{z}_1} = \frac{\left(\tilde{z}_1 - \frac{1}{2}\right)\pi}{n_k}\left(1 - \frac{h_{n_k}}{n_k}\right) + o\left(\frac{1}{n_k}\right)$$

From this, it follows that:

$$j(n_k)\left(1-\frac{h_{n_k}}{n_k}\right)+o(1)=\tilde{z}_1\left(1-\frac{h_{n_k}}{n_k}\right)+o(1)$$

which implies that for all sufficiently large$t$, $j(n_k)=\tilde{z}_1$. Therefore, we conclude that:

$$x_{n_k}^{j(n_k)}=x_{\tilde{n}_t}^{j(\tilde{n}_t)}$$

Lemma 3. ([40])
The nodal lengths corresponding to the boundary value problem L satisfy the following asymptotic approximations:
For nodal points x_n^j lying in the interval $(0,a)$ we have:

$$\ell_n^{(j)}=\frac{\pi}{\lambda_n^0}+\frac{1}{\left(\lambda_n^0\right)^2}\int_{x_n^{j(n)}}^{x_n^{j(n)+1}}q(t)\,dt+o\left(\frac{1}{\left(\lambda_n^0\right)^2}\right) \tag{13}$$

Theorem 2. ([40]). Let $x\in(0,a)$, $\mathfrak{R}=\left\{r\pi:\ r=\frac{p}{q},\ p<q\ ,\ p,q\in\mathrm{N}\right\}$and assume $x_k^{(n_k)}\subset X(L)$ with $\lim\limits_{k\to\infty}x_k^{(n_k)}=x$. Then, for any point $a\in\mathfrak{R}$the following limits exist and are finite:

$$f_1(x):=\lim_{k\to\infty}\left[n_kx_{n_k}^{j(n_k)}-\left(j(n_k)-\frac{1}{2}\right)\pi\right],x_{n_k}^{j(n_k)}\in(0,a)$$

$$g_1(x):=\lim_{k\to\infty}n_k\left[n_kx_{n_k}^{j(n_k)}-\left(j(n_k)-\frac{1}{2}\right)\pi+\frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k}\right],x_{n_k}^{j(n_k)}\in(0,a)$$

$$f_2(x):=\lim_{k\to\infty}\left[n_kx_{n_k}^{j(n_k)}-\left(j(n_k)-\frac{1}{2}\right)\pi\right],x_{n_k}^{j(n_k)}\in(a,\pi)$$

$$g_2(x):=\lim_{k\to\infty}n_k\left[n_kx_{n_k}^{j(n_k)}-\left(j(n_k)-\frac{1}{2}\right)\pi-\arctan\left(\frac{B_{n_k}}{A_{n_k}}\right)\frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k}h_{n_k}\right],x_{n_k}^{j(n_k)}\in(a,\pi).$$

Theorem 3. ([40]). Let $X_0(L)\subset X(L)$) be a subset of nodal points that is dense in the interval$(0,\pi)$. Then, for any real number $a\in\mathrm{R}$the specification of the set $X_0(L)$uniquely determines the potential function $q(x)-\langle q\rangle$ almost everywhere on $(0,\pi)$as well as the boundary coefficientsh,H and the interface parameterβ.
Moreover, the potential $q(x)-\langle q\rangle\rangle$, along with the constants h,H and βcan be reconstructed through the following procedure:

1. For each $x\in(0,\pi)$), select a sequence $\left\{x_{n_k}^{j(n_k)}\right\}\subset X_0(L)$such that

$$\lim_{n\to\infty}x_n^{j(n)}=x$$

$$h=g_1(0)$$

$$\beta=(1+\tan ah_0)\tan\left(\frac{f(a^+)-f(a^-)}{2}\right)$$

Here, the function $f(x)$is defined piecewise as:

$$f(x)=\begin{cases} f_1(x)\,,\ x\in[0,a)\\ f_2(x)\,,x\in(a,\pi]\end{cases}$$

1. The function $q(x) - \langle q \rangle$ can be explicitly reconstructed using the expression:

$$q(x) - \langle q \rangle = 2g_1(x) + \frac{2(h+H)\left[\sin \pi h_0 + \frac{1}{2}\cos \pi h_0\right] - \frac{1}{2}\beta (H-h)\cos(2a-\pi)h_0}{\pi\left[\cos \pi h_0 + \frac{1}{2}\sin \pi h_0\right] + \frac{1}{2}\beta(2a-\pi)\sin(2a-\pi)h_0}$$

$$\begin{aligned}\langle q \rangle = & \tfrac{1}{\left[\cos \pi h_0 + \frac{1}{2}\sin \pi h_0\right] + \frac{1}{2}\beta(2a-\pi)\sin(2a-\pi)h_0} \Big\{\Big[-\tfrac{1}{2}\left(\beta \sin \pi h_0 + \cos \pi h_0\right. \\ & \left.+\tfrac{1}{2}\beta\cos(2a-\pi)h_0\right) + \tfrac{1}{2}\left(1-\beta^2\right)\sin(2a-\pi)h_0\Big]\int_0^a q(t)\,dt \\ & +\Big[\sin \pi h_0 + \beta\cos \pi h_0 + \tfrac{1}{2}\beta\cos(2a-\pi)h_0\Big]\int_0^a q(t)\,dt\Big\}\end{aligned}$$

This formulation expresses how the mean value of the potential function is analytically related to the boundary parameters and the discontinuity coefficient β.

3. Numerical Strategy for Solving the Inverse Problem

Inverse Problem Statement.
Given a set of nodal points $\left\{x_n^{j(n)}\right\}$ for $j = 1, 2, ..., n-1$ and $n \in \mathrm{N}$ the objective is to reconstruct the potential function $q(x)$.
Since the nodal values $\left\{x_n^j\right\})\}$ represent the zeros of the nnnth eigenfunction $\varphi(x, \lambda_n)$, one can express the inverse problem accordingly as follows:

$$\begin{cases} \varphi\left(x_n^j, \lambda_n\right) = 0, x_n^j \in \left(0, \frac{\pi}{2}\right), n = 2m \\ \varphi\left(x_n^j, \lambda_n\right) = 0, x_n^j \in \left(\frac{\pi}{2}, \pi\right), n = 2m \end{cases}$$

By replacing φ_n with $\varphi_n\left(x_n^{j(n)}\right)$, the expression becomes:

$$\int_0^{x_n^j} q(t)\sin \lambda_n^0\left(x_n^j - t\right)\cos(\lambda_n t)\,dt \cong -h\sin\left(\lambda_n^0 x_n^j\right) - \lambda_n^0\cos\left(\lambda_n^0 x_n^j\right)$$

$$\sin \lambda_n^0 x_n^j \int_0^{x_n^j} q(t)\cos^2(\lambda_n t)\,dt - \frac{\cos\left(\lambda_n^0 x_n^j\right)}{2}\int_0^{x_n^j} q(t)\sin 2\left(\lambda_n^0 t\right)dt \cong -h\sin\left(\lambda_n^0 x_n^j\right) - \lambda_n^0\cos\left(\lambda_n^0 x_n^j\right)$$

Dividing both sides of the expression by $\sin\left(\lambda_n^0 x_n^j\right)$, we arrive at

$$\int_0^{x_n^j} q(t)\cos^2(\lambda_n t)\,dt \cong -h - \lambda_n^0\cot\left(\lambda_n^0 x_n^j\right).$$

Let$f \in L_2(0, \pi)$. Then $f(t)$can be expressed in the form

$$f(t) = \sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty} y_{\ell_m}\psi_{\ell_m}(t)$$

where the collection $\{\psi_{\ell m}(t)\}$constitutes an orthogonal system in $L_2(0, \pi)$. Therefore, the coefficients $y_{\ell m}$ can be interpreted as the Fourier coefficients of $f(t)$ with respect to this basis or as projections onto these basis functions:

$$y_{\ell m} = \int_0^{\pi} f(t)\,\psi_{\ell m}(t)\,dt$$

We may write the expansion compactly as:
$f(t) \cong \sum_{\ell=1}^{2^{k-1}}\sum_{m=0}^{M-1} y_{\ell m}\psi_{\ell m}(t) = Y^T \cdot \psi(t)$
where the coefficient vector $Y$and the basis vector$\psi(t)$ are defined respectively as:

$$Y = \left[y_{10}\ y_{11}\ y_{12} \dots y_{1(M-1)}\ y_{20}\ y_{21} \dots y_{2^{k-1}0}\ y_{2^{k-1}1}\ y_{2^{k-1}2} \dots y_{2^{k-1}(M-1)}\right]^T$$

$$\psi(t) = \left[\psi_{10}\ \psi_{11}\ \psi_{12} \ldots \psi_{1(M-1)}\ \psi_{20}\ \psi_{21}\ \ldots \psi_{2^{k-1}0}\ \psi_{2^{k-1}1}\ \psi_{2^{k-1}2} \ldots \psi_{2^{k-1}(M-1)}\right]$$

The basis function $\psi_{\ell m}(t)$is defined piecewise as:

$$\psi_{\ell m}(t) = \begin{cases} \frac{2^{\frac{k}{2}}}{\pi}\tilde{L}_m\left(\frac{2^k t}{\pi} - 2c + 1\right), & if\ \frac{(2c-2)\pi}{2^k} \le t < \frac{2c\pi}{2^k} \\ 0 & , \quad otherwise \end{cases}$$

where $\ell = 1, \ldots, 2^{k-1}$ and

$$\tilde{L}_m(t) = \sqrt{m + \frac{1}{2}} \cdot L_m(t),\ m = 0, 1, \ldots M - 1$$

with $L_m(t)$ denoting the classical Legendre polynomials.
Let the first two Legendre polynomials be defined as:

$$L_0(t) = 1\ ,\ L_1(t) = t$$

for $m = 1, 2, \ldots$, the higher-order polynomials can be generated recursively using:

$$L_{m+1}(t) = \frac{2m+1}{m+1} t \cdot L_m(t) - \frac{m}{m+1} \cdot L_{m-1}(t)$$

Since the function $q(x)$belongs to the space $L_2(0, \pi)$, it can be expanded in terms of the orthogonal basis functions $\psi_{\ell m}(t)$, yielding the following representation:

$$q(t) = \sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty} y_{\ell m}\psi_{\ell m}(t) \cong \sum_{\ell=1}^{2^{k-1}}\sum_{m=0}^{M-1} y_{\ell m}\psi_{\ell m}(t)$$

$$\sum_{\ell=1}^{2^{k-1}}\sum_{m=0}^{M-1} y_{\ell m}\left\{\int_0^{x_n^j} \psi_{\ell m}(t)\sin\lambda_n^0\left(x_n^j - t\right)\cos\lambda_n^0 t dt\right\} \cong -h\sin\lambda_n^0 x_n^j - \lambda_n^0\cos\lambda_n^0 x_n^j$$

and this holds for $n > 1$, $j = 1, \ldots, n$.
Let us now proceed with the numerical algorithm.

1. Suppose the nodal points $\left\{x_n^j\right\}$ for $n > 1$and $j = 1, 2, \ldots, n$ are provided. Let $k, M, h, H \in$ R, and choose $n = 2^{k-1} \cdot m$.
2. The vector Y is determined by the linear system $AxY = B$, where:

$$A = [A_1\ A_2\ A_3 \ldots .\ A_{2^{k-1}}\ ]$$

with each A_ℓ given by:

$$A_\ell = \begin{pmatrix} \int_0^{x_n^1} \psi_{\ell_0}\sin\lambda_n^0\left(x_n^1 - t\right)\cos(\lambda_n t)\,dt & \ldots & \int_0^{x_n^1} \psi_{\ell_{(M-1)}}\sin\lambda_n^0\left(x_n^1 - t\right)\cos(\lambda_n t)\,dt \\ \int_0^{x_n^2} \psi_{\ell_0}\sin\lambda_n^0\left(x_n^2 - t\right)\cos(\lambda_n t)\,dt & \ldots & \int_0^{x_n^2} \psi_{\ell_{(M-1)}}\sin\lambda_n^0\left(x_n^2 - t\right)\cos(\lambda_n t)\,dt \\ \\ \int_0^{x_n^n} \psi_{\ell_0}\sin\lambda_n^0(x_n^n - t)\cos(\lambda_n t)\,dt & \ldots & \int_0^{x_n^n} \psi_{\ell_{(M-1)}}\sin\lambda_n^0(x_n^n - t)\cos(\lambda_n t)\,dt \end{pmatrix}$$

The right-hand side vector $B$is defined as:

$$B = \left[-h\sin\left(\lambda_n^0 x_n^1\right) - \lambda_n^0\cos\left(\lambda_n^0 x_n^1\right) \quad \ldots \quad -h\sin\left(\lambda_n^0 x_n^n\right) - \lambda_n^0\cos\left(\lambda_n^0 x_n^n\right)\right]^T$$

1. To compute the values $q(t_i)$for $i = 1, 2, \ldots, m$, we use the system:

$$[q(t_i)] = Y^T x\phi$$

where the sample points t_i are defined by

$$t_i = \frac{(2i-1)\,\pi}{2^k \cdot M}, i = 1, 2, ..., 2^{k-1} \cdot M = M'$$

and the vector $\phi$is given by:

$$\phi = \left[\psi\left(\frac{\pi}{2M'}\right) \;\; \psi\left(\frac{3\pi}{2M'}\right) \quad .... \quad \psi\left(\frac{(2M'-1)\,\pi}{2M'}\right)\right].$$

Accordingly, the potential function $q\,(t)$can be represented as an infinite series:

$$q\,(t) = \sum_{i=1}^{\infty} q_i\,(t)$$

where each term $q_i\,(t)$ is expressed as:

$$q_i = \sum_{m=0}^{\infty} y_{\ell_m} \psi_{\ell_m}.$$

4. Experimental Analysis and Numerical Validation

In this part, we demonstrate several computational examples to illustrate the application of the inverse nodal problem using the previously outlined algorithm. All visualizations and computations are carried out with the help of MATLAB. These examples clearly reveal the convergence behavior of the suggested approach and highlight the stability of the reconstructed solution to the inverse problem.

Example 1.
Let us consider the function $q\,(x) = \cos 2x$
We take the boundary parameters as $h = 1$, $H = 2$, $\beta = 1$and let $a = \frac{\pi}{2}$.
Assuming $k = 4$ and $m = 3$, it follows that $n = 2^{k-1} \cdot m = 24$. Therefore,$j = 1, 2, ..., 24$.
Under these conditions, the nodal points satisfy the following relations:
$\frac{1}{3}\left(x_{24}^{(j)}\right)^3 + \left(1 - \frac{2}{7^4} + \frac{12}{7^4 \cdot \pi}\right)\left(x_{24}^{(j)}\right) \cong \frac{(2j-1)\cdot\pi}{49} + \frac{4}{7^4}(j = 1, 2, ..., 24),\, a \in \left(0, \frac{\pi}{2}\right)$

$$\frac{4}{7^3 \cdot 3}\left(x_{24}^{(j)}\right)^3 + \left(1 - \frac{2}{7^3} + \frac{36}{7^4 \cdot \pi}\right)\left(x_{24}^{(j)}\right) \cong \frac{(2j-1)\cdot\pi}{49} + \frac{\pi}{98} + \frac{12}{7^4}\,(j = 1, 2, ..., 24)\,, a \in \left(\frac{\pi}{2}, \pi\right)$$

These expressions define the roots of the equations. The numerical values of these roots are given in Table 1.

4.1. *Table 1. Numerical values of nodal points in Example 1*

j	1	2	3	4	5
x_{24}^{j}	0.10127225	0.22959331	0.35782623	0.48592215	0.61383272
j	6	7	8	9	10
x_{24}^{j}	0.74151030	0.86890811	0.99598043	1.12268269	1.24897168
j	11	12	13	14	15
x_{24}^{j}	1.37480562	1.50014431	1. 6249492	1.74918353	1.87281231
j	16	17	18	19	20
x_{24}^{j}	1.99580248	2.11812286	2.23974423	2.36063934	2.48078290
j	21	22	23	24	
x_{24}^{j}	2. 60015156	2.71872391	2.83648045	2.95340354	

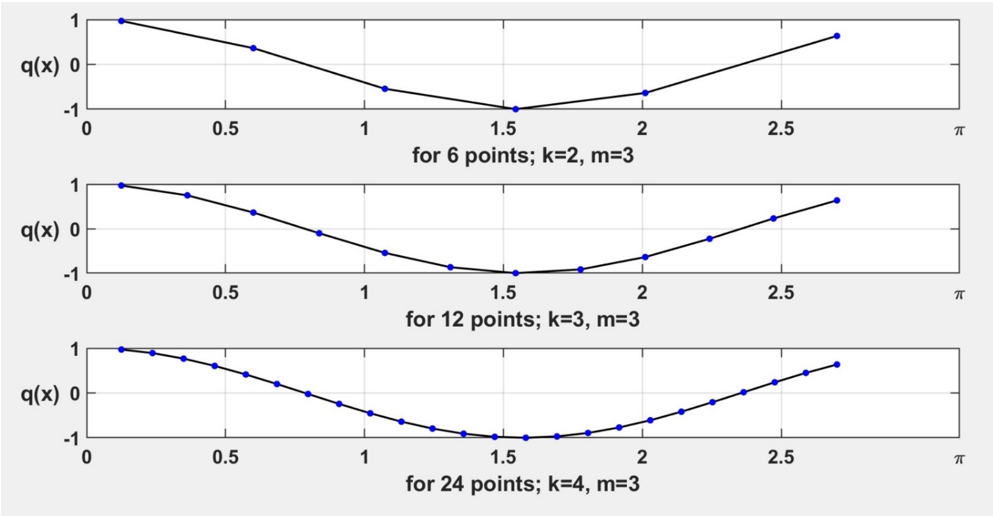

Figure 1: Approximate reconstruction of the potential function $q(x) = \cos 2x$ using 6, 12, and 24 nodal points.

At this stage, we treat q as an unknown potential function, and assume that the nodal data listed in Table 1 serve as the input for our computations. Utilizing the proposed algorithm, our objective is to approximate the function q as the solution to the associated inverse nodal problem.

Example 2.
Let us consider the function $q(x) = 4x^3 - 2x$.
We take the boundary parameters as $h = 1$, $H = 2$and $\beta = 1$and let $a = \frac{\pi}{2}$.
Assuming $k = 4$ and $m = 3$, it follows that $n = 2^{k-1} \cdot m = 24$. Therefore,$j = 1, 2, ..., 24$.
Under these conditions, the nodal points satisfy the following relations:

$$-\frac{2}{7^4}\left(x_{24}^{(j)}\right)^4 + \frac{2}{7^4}\left(x_{24}^{(j)}\right)^2 + \left(1+\frac{4}{7^4}\left(\frac{5}{\pi}+\frac{15\pi^3}{8}-\frac{3\pi}{2}\right)\right)\left(x_{24}^{(j)}\right) \cong \frac{(2j-1)\pi}{49}+\frac{4}{7^4}, (j=1,2,...,24), a\in\left(0,\frac{\pi}{2}\right)$$

$$-\frac{7}{2}\left(x_{24}^{(j)}\right)^4 - \left(x_{24}^{(j)}\right)^2 + \left(\frac{20}{\pi}+\frac{15\cdot\pi^3}{2}-6\cdot\pi+\frac{(49)^2}{4}\right)\left(x_{24}^{(1)}\right) \cong 4-\frac{5\cdot\pi^4}{12}+\frac{5\cdot\pi^2}{8}+\frac{(2j-1)\cdot 49\cdot\pi}{4}+\frac{49\cdot\pi}{8}(j=1,2,...,24)\, a\in\left(\frac{\pi}{2},\pi\right)$$

4.2. *Table 2. Numerical values of nodal points in Example 2*

j	1	2	3	4	5
x_{24}^j	0.033285160	0.127050880	0.220749620	0.314396110	0.408012880
j	6	7	8	9	10
x_{24}^j	0.501632020	1.6242536421	0.595286280	0.689027260	0.876990170
j	11	12	13	14	15
x_{24}^j	0.971350130	1.066068160	1.161239990	1.256973590	1.353392140
j	16	17	18	19	20
x_{24}^j	1.450636720	1.548869800	1.648279900	1.749087970	1.851559490
j	21	22	23	24	
x_{24}^j	1.955984040	2.062790590	2.172416520	2.285476130	

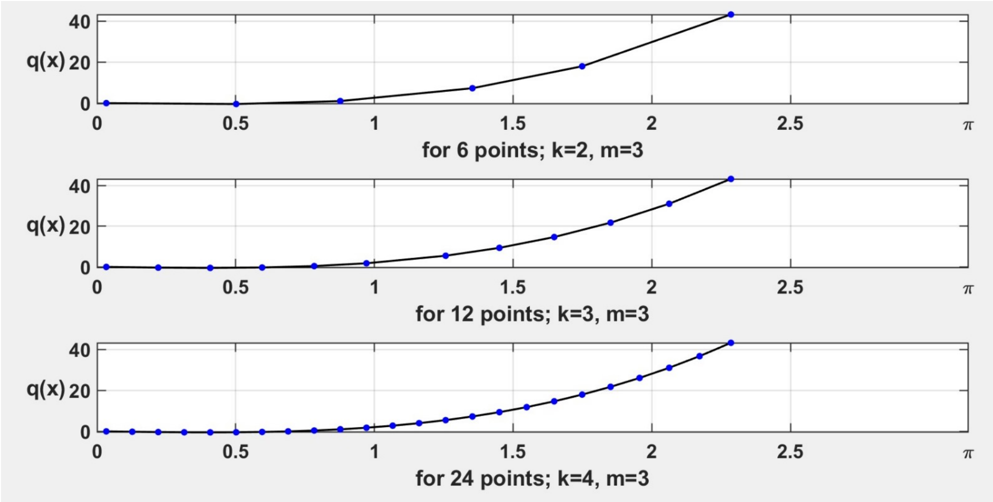

Figure 2: Approximate reconstruction of the potential function $q(x) = 4x^3 - 2x$ using 6, 12, and 24 nodal points.

For instance, as revealed by the behavior of the nodal points $x_n^{j(n)}$ within the interval $\left(0, \frac{\pi}{2}\right)$, the values j that satisfy the condition $x_n^{j(n)} > \frac{\pi}{2}$ are selected to ensure the given inequality holds, and these values coincide with the value of $x_n^{j(n)}$ in the interval $\left(\frac{\pi}{2}, \pi\right)$.

Conclusion

As indicated by the asymptotic behavior of the nodal points $x_n^{j(n)}$, if no discontinuity conditions exist at the point $x = \frac{\pi}{2}$, then the behavior of all nodal points of the problem coincides with their behavior in the interval $\left(0, \frac{\pi}{2}\right)$. Consequently, if discontinuity conditions are indeed present at $x = \frac{\pi}{2}$, the behavior within $\left(0, \frac{\pi}{2}\right)$ must coincide with that in the interval $\left(\frac{\pi}{2}, \pi\right)$ for sufficiently large values of j. Therefore, although the behaviors of the nodal points $x_n^{j(n)}$ in the intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ may appear different, their positions can be adjusted within both intervals depending on the values of j. Hence, when plotting the potential function with respect to the nodal points $x_n^{j(n)}$, it is essential to utilize the behavior observed in both intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. Based on this idea, Graph 1 and Graph 2 have been constructed.

Subsequently, we demonstrated the stability of the inverse nodal formulation. Moreover, an approximate solution was developed using the Chebyshev interpolation method tailored to the Sturm-Liouville operator with jump discontinuities. Employing Chebyshev polynomials of the first kind, we provided computational examples that illustrate the numerical reconstruction of the potential function and confirmed the stability of the results.

Recent advancements have shown the Chebyshev interpolation technique to be particularly effective in the numerical treatment of integral and integro-differential equations. Within the context of this study, it has been established that this approach is also well-suited for computing approximate solutions to inverse nodal problems with discontinuities.

References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Kron, and H. Holden, Solvable models in quantum mechanics, Springer, New York-Berlin, 1988, pp. xiv+452.

[2] R. K. H. Amirov and I. M. Guseinov, Boundary value problems for a class of Sturm-Liouville operator with nonintegrable potential, Differ. Equ. 38 (2002), no. 8, 1195–1197)

[3] M. M. Dzh, Inverse spectral and inverse nodal problems for Sturm-Liouville equations with point - interactions, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerbaijan 45 (2019), no. 2, 286–294.

[4] M. Savchuk, On the eigenvalues and eingenfunctions of the Sturm-Liouville operator with singular potential, Math. Notes 69 (2001), 2.
[5] Ch.G. Ibadzadeh, L. I.Mammadova, and I. M.Nabiev, Inverse problem of spectral analysis for diffusion operatorwith nonseparated boundary conditions and spectral parameter in boundary condition, Azerbaijan J. Math. 9 (2019), no. 1, 171–189.
[6] I. M. Nabiev, Reconstruction of the differential operator with spectral parameter in the boundary condition, Mediterr. J. Math. 19 (2022), no. 3, 124.
[7] L. I. Mammadova and I. M. Nabiev, Rzayeva Ch.H. Uniqueness of the solution of the inverse problem for differential operator with semiseparated boundary conditions, Baku Math. J. 1 (2022), no. 1, 47–52.
[8] M. G. Gasymov and G. S. h. Guseinov, Determination of a diffusion operator from spectral data, In Dokl.Akad. Nauk Azerb SSR 37 (1981), no. 2, 19–23.
[9] E. Yilmaz, H. Koyunbakan, and S. Ambarzumyan, Type theorems for bessel operator on a finite interval, Differ. Equ. Dyn. Syst. 27 (2019), 553–559.
[10] S.C. Bartels, S. Currie, and B. A.Watson, Sturm-liouville problems with transfer conditionHerglotz dependent on the eigenparameter: eigenvalue asymptotics, Complex Anal, Oper. Theory 15 (2021), 71.
[11] M. A. Kuznetsova, On recovering the Sturm-Liouville differential operators on time scales, Math. Notes 109 (2021), 74–88.
[12] P. Bondarenko and E. E. Chitorkin, Inverse Sturm-Liouville problem with spectral parameter in the boundary conditions, Mathematics 11 (2023), no. 5, 1138.
[13] G. Du, C. Gao, and J. Wang, Spectral analysis of discontinuous Sturm-Liouville operators with Herglotzs transmission, Electron. Res. Arch. 31 (2023), no. 4, 2108–2119.
[14] N. J. Guliyev, Inverse square singularities and eigenparameter-dependent boundary conditions are two sides of the same coin, 2023, DOI 10.1093/qmath/haad004.
[15] N. J. Guliyev, Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter, J. Math. Phys. 60 (2019), no. 6, 063501.
[16] N. J. Guliyev, Essentially isospectral transformations and their applications, Ann. Mat. Pura Appl. 199 (2020), no. 4, 1621–1648.
[17] N. J. Guliyev, On two-spectra inverse problems, Proc. Amer. Math. Soc. 148 (2020), no. 10, 4491–4502. no. 2, 19–23.
[18] J. R. McLaughlin, Inverse spectral theory using nodal points as data - a uniqueness result, J. Differ. Equ. 73 (1988), 354–362..
[19] O. H. Hald and J. R. McLaughlin, Solutions of inverse nodal problems, Inverse Probl. 5 (1989), 307–347
[20] C. F. Yang, A solution of the nodal problem, Inverse Probl. 13 (1997), 203–213.
[21] S. A. Buterin and S. T. Chung, Inverse nodal Problem for differential pencils, Appl. Math. Lett. 22 (2009), 1240–1247.
[22] P. J. Browne and B. D. Sleeman, Inverse nodal problem for Sturm–Liouville equation with eigenparameter depend boundary conditions, Inverse Probl. 12 (1996), 377–381.
[23] S. A. Buterin and C. T. Shieh, Incomplete inverse spectral and nodal problems for differential pencil, Results Math. 62 (2012), 167–179..
[24] Y. H. Cheng, C. K. Law, and J. Tsay, Remarks on a new inverse nodal problem, J. Math. Anal. Appl. 248 (2000), 145–155.
[25] L.S. Currie and B. A.Watson, Inverse nodal problems for Sturm–Liouville equations on graphs, Inverse Probl. 23 (2007), 2029–2040..
[26] Y. X. Guo and G. S.Wei, Inverse problems: Dense nodal subset on an interior subinterval, J. Differ. Equ. 255 (2013), 2002–2017.
[27] C. K. Law and J. Tsay, On the well-posedness of the inverse nodal problem, Inverse Probl. 17 (2001), 1493–1512..
[28] A. S. Ozkan, Half inverse Sturm?Liouville problem with boundary and discontinuity conditions dependent on the spectral parameter, Inverse Probl. Sci. Eng. 22 (2014), 848–859.
[29] . C. L. Shen and C. T. Shieh, An inverse nodal problem for vectorial Sturm–Liouville equation, Inverse Probl. 16 (2000), 349–356..
[30] C. T. Shieh and V. A. Yurko, Inverse nodal inverse spectral problems for discontinuous boundary value problems, J. Math. Anal. Appl. 347 (2008), 266–272.
[31] C. F. Yang, A new inverse nodal problem, J. Differ. Equ. 169 (2001), 633–653.
[32] C. F. Yang and X. P. Yang, Inverse nodal problems for the Sturm–Liouville equation with polynomially dependent on the eigenparameter, Inverse Probl. Sci. Eng. 19 (2011), 951–961.
[33] C. F. Yang, Inverse nodal problems of discontinuous Sturm–Liouville operator, J. Differ. Equ. 254 (2013), 1992–2014.
[34] H. Koyunbakan and S. Mosazadeh, Inverse nodal problem for diccontinuous Sturm–Liouville operator by new prüfer substitutions, Math. Sci. 15 (2021), no. 4, 387–394.
[35] C. F.Yang andD. Q. Liu, Inverse nodal problems on quantum tree graphs, Proc. Royal Soc. of Edinburgh Sect. A. Math. 153 (2023), 275–288. no. 2, 19–23.
[36] C. F. Yang, Stability in the inverse nodal solution for the interior transmission problem, J. Differ. Equ. 260 (2016), no. 3, 2490–2506.
[37] Y. T. Hu, N. P. Bondarenko, C. T. Shieh, and C. F. Yang, Traces and inverse nodal problems for Dirac-type integro-differential operators on a graph, Appl. Math. Comput. 363 (2019), 124606.
[38] R. Zhang, M. Sat, and C. F. Yang, Inverse nodal problem for the Sturm–Liouville operator with a weight, Appl. Math. J. Chinese Univ. 35 (2020), no. 2, 193–202.
[39] I. M. Guseinov and L. I. Mammadova, Properties of the eigenvalues of the Sturm-Liouville operator with discontinuity conditions inside the interval, Pross. Baku State University, phys-math.sci. series (2011), 3..
[40] R. K. h. Amirov, and S. Durak, Inverse nodal solution for singular diffusion equation, J. Math. Meth. Appl. Sci. , (2024), no. 47, 9067-9083. no. 2, 19–23.
[41] I. M. Guseinov and L. I. Mammadova, Reconstruction of the diffusion equation with singular coefficients for two spectra, Doklady Mathematics 90 (2014), no. 1, 401–404.
[42] . A. Nabiev and R. K. h. Amirov, Integral representations for the solutions of the generalized Schroedinger equation in a finite interval, 2015. Advances in Pure Mathematics.
[43] V. A. Marchenko, Sturm-Liouville Operators and Their Applications (1977), 1986.
[44] L. B. Ya, Lectures on Entire Functions, Transl. Math Monographs, Vol. 150, Amer. Math. Soc. Providence RI, 1996.