

The Fractured Continuum: Non-Archimedean Spacetime and Field Dynamics in Post-Quantum Calculus

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Abstract: In this study, we present a new formulation of quantum field dynamics by replacing the classical continuum with the algebraic structure of (p, q) -calculus. By introducing a dual-parameter deformation scale where $0 < q < p \leq 1$, we construct an invariant field theory that inherently lacks the infinitesimals of standard calculus. This work also demonstrates that the unanchored (p, q) -difference operator introduces a natural, non-local quantum fuzziness that effectively regularizes ultraviolet (UV) divergences without artificial momentum cutoffs. We derive the generalized (p, q) -Hermite-Hadamard bounds to constrain particle localization paths. Finally, we establish a new multi-scale bound proving that all physical states within this non-Archimedean spacetime framework are immune to infinite derivative singularities.

Keywords: (p, q) -Calculus, Non-Archimedean analysis, Hermite-Hadamard inequalities, Discrete Difference operators, Planck Scale geometry, Ultraviolet divergence, Quantum action confinement.

Symbol	Definition
$\mathcal{M}_{p,q}$	Fractured spacetime continuum
J_a	Localized spacetime interval
$[n]_{p,q}$	Quantum integer (p, q) -bracket
$[n]_{p,q}!$	(p, q) -factorial
$\binom{n}{k}_{p,q}$	(p, q) -binomial coefficient
$D_{p,q}$	Unanchored (p, q) derivative
$\int_a^b \dots d_{p,q}t$	Definite anchored $(p, q)_a$ - integral
$\psi(x)$	Quantum state field function
$e_{p,q}(z)$	(p, q) -exponential function
$V(x)$	Spatial potential energy distribution
k	Momentum scale factor
Λ_{UV}	Ultimate geometric ultraviolet cutoff
L	Macroscopic Lipschitz continuity constant
x, t	Spacetime coordinate elements

of quantum fluctuations and gravitational singularity.

While classical quantum calculus historically bridged this gap by evaluating discrete differences without limits where taking the limit $q \rightarrow 1^-$ recovers classical math [8],[11] it remains inherently asymmetric. Post-quantum calculus or (p, q) -calculus generalizes this by introducing a secondary scale parameter, p , decoupling the scale grid from the origin and offering a symmetric, multi-scale description of discrete spacetime [6], [3]. When $p = 1$, the framework collapses gracefully back into q -calculus [1], [4].

This paper formalizes a groundbreaking field theory built completely upon $(\mathbb{N}_p, q)$ -differential and integral identities, uncovering a quantum mechanical geometry that is inherently discrete and free of the structural infinite values that plague standard quantum field theories.

1 Introduction

For centuries, physics has relied on the foundational work of Newtonian and Leibnizian calculus where the whole system is tethered to the limit-based continuum $\lim_{\Delta x \rightarrow 0}$. However, at the Planck scale, the physical meaning of an arbitrary small spatial interval collapses under the weight

2 Definitions and Notations

Throughout this framework, spacetime is treated as a multi-scale deformed geometric lattice denoted by $\mathcal{M}_{p,q}$.

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2.1 Deformation Parameters (p, q) : Two fixed, dimensionless real numbers dictating the quantization scale of the continuum, strictly bounded by $0 < q < p \leq 1$ [6].

2.2 The Spacetime Interval (J_a) : A localized chunk of space or time anchored at a topological reference point $a \in \mathbb{R}$, defined as $J_a = \{a + (b-a)p^m q^n : m, n \in \mathbb{Z}\}$ [9].

2.3 For any non-negative integer $n \in \mathbb{N}_0$, the (p, q) -bracket

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + q^{n-2}p + \dots + pq^{n-2} + q^{n-1} \quad (1)$$

2.4 The (p, q) -factorial operator tracks the product of these quantum integers:

$$[n]_{p,q}! = \begin{cases} \prod_{k=1}^n [k]_{p,q}, & n \geq 1 \\ 1, & [0]_{p,q}! = 1 \end{cases} \quad (2)$$

2.5 The (p, q) -Operators and Special Functions

Let $\psi : J_a \rightarrow \mathbb{R}$ be a continuous function. The anchored (p, q) -derivative evaluates field differences relative to the coordinate anchor a [2]:

$$D_{p,q}\psi(x) = \frac{\psi(px) - \psi(qx)}{(p-q)x}, \quad x \neq a \quad (3)$$

2.6 The (p, q) -exponential function [12]:

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} z^n}{[n]_{p,q}!} \quad (4)$$

3 Main Result

We now establish the mathematical framework of the Fractured Continuum through three core theorems, demonstrating how this geometry naturally transforms quantum behaviors.

3.1 Theorem 1: Ultraviolet Self-Regularization

Let $\psi(x) = e_{p,q}(ikx)$ represent a high momentum plane wave field configuration with momentum scale k within the fractured continuum $\mathcal{M}_{p,q}$. As the momentum scales deep into the ultraviolet regime $k \rightarrow \infty$, the field's kinetic energy density $|D_{p,q}\psi(x)|^2$ remains strictly bounded by an absolute geometric upper bound Λ_{UV} , preventing ultraviolet divergences without requiring an artificial momentum cutoff.

Proof.

In classical calculus, the kinetic energy density of a plane wave scales continuously as $|\partial_x e^{ikx}|^2 = k^2$, which yields a divergent infinity as $k \rightarrow \infty$. In the fractured continuum, we calculate the unanchored (p, q) -derivative of our field configuration

$$D_{p,q}e_{p,q}(ikx) = \frac{e_{p,q}(ipkx) - e_{p,q}(iqkx)}{(p-q)x} \quad (5)$$

$$D_{p,q}(ikx)^n = \frac{(ikpx)^n - (ikqx)^n}{(p-q)x} \quad (6)$$

$$= (ik)^n x^{n-1} \left(\frac{p^n - q^n}{p - q} \right) \quad (7)$$

$$= (ik)^n x^{n-1} [n]_{p,q} \quad (8)$$

Substituting this back into the derivative yields:

$$D_{p,q}e_{p,q}(ikx) = \sum_{n=1}^{\infty} \frac{(ik)^n x^{n-1} [n]_{p,q}}{[n]_{p,q} [n-1]_{p,q}!} \quad (9)$$

$$= i(x)k \sum_{n=1}^{\infty} \frac{(ikx)^{n-1}}{[n-1]_{p,q}!} \quad (10)$$

$$= ike_{p,q}(ikx) \quad (11)$$

Evaluating the magnitude of the difference operator to find the kinetic energy density bounds:

$$|D_{p,q}\psi(x)|^2 = \left| \frac{\psi(px) - \psi(qx)}{(p-q)x} \right|^2 \quad (12)$$

Since $|\psi(x)| \leq 1$, applying the triangle inequality to the numerator gives:

$$|D_{p,q}\psi(x)|^2 \leq \left(\frac{|\psi(px)| + |\psi(qx)|}{(p-q)|x|} \right)^2 \leq \left(\frac{1+1}{(p-q)x} \right)^2 \quad (13)$$

$$= \frac{4}{(p-q)^2 x^2} \quad (14)$$

$$\Lambda_{UV} = \frac{4}{(p-q)^2 x^2} \quad (15)$$

As $k \rightarrow \infty$, physical infinity is avoided. The non-Archimedean spacing creates a natural lattice dampening effect built into the calculus itself.

3.2 Theorem 2: Spacetime Quantum Confinement

Let $V : [a, b] \rightarrow \mathbb{R}$ be a convex potential energy field configuration across a fractured spacetime interval. The total integrated action of the field within this quantum chunk is strictly trapped between its spatial midpoints and boundary conditions, proving that quantum paths are confined to stable trajectories and cannot fluctuate into arbitrary chaos.

Proof.

By the definition of a convex energy, for any real parameter $t \in [0, 1]$:

$$V(tb + (1-t)a) \leq tV(b) + (1-t)V(a) \quad (16)$$

Analyzing the definite anchored $(p, q)_a$ -integral of V over $[a, pb + (1-p)a]$ (Kunt et al., 2021):

$$\begin{aligned} \int_a^{pb+(1-p)a} V(x)_a d_{p,q}x &= (p-q)(b-a)p \\ &\times \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \\ &\times V\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right). \end{aligned} \quad (17)$$

Setting $t = \frac{q^n}{p^n}$

$$V\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \leq \frac{q^n}{p^n}V(b) + \left(1 - \frac{q^n}{p^n}\right)V(a) \quad (18)$$

Substituting back gives:

$$\begin{aligned} \int_a^{pb+(1-p)a} V(x)_a d_{p,q}x &\leq (p-q)(b-a)p \\ &\times \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \\ &\times \left[\frac{q^n}{p^n}V(b) + \left(1 - \frac{q^n}{p^n}\right)V(a) \right]. \end{aligned} \quad (19)$$

Evaluating the $V(b)$ co-factor series:

$$\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \cdot \frac{q^n}{p^n} = \frac{1}{p} \sum_{n=0}^{\infty} \left(\frac{q^2}{p^2}\right)^n \quad (20)$$

Since $0 < q < p \leq 1$, the common ratio satisfies $|r| = \frac{q^2}{p^2} < 1$. Using the sum to infinity geometric progression $\frac{1}{1-r}$:

$$\frac{1}{p} \left(\frac{1}{1 - \frac{q^2}{p^2}} \right) = \frac{1}{p} \left(\frac{p^2}{p^2 - q^2} \right) = \frac{p}{p^2 - q^2} \quad (21)$$

Evaluating the $V(a)$ co-factor series:

$$\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n}{p^n}\right) = \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} - \sum_{n=0}^{\infty} \frac{q^{2n}}{p^{2n+1}} \quad (22)$$

When the ratio is $\frac{q}{p}$

$$\frac{1}{p} \left(\frac{1}{1 - \frac{q}{p}} \right) = \frac{1}{p} \left(\frac{p}{p-q} \right) = \frac{1}{p-q} \quad (23)$$

$$\frac{1}{p-q} - \frac{p}{p^2 - q^2} = \frac{p+q}{p^2 - q^2} - \frac{p}{p^2 - q^2} = \frac{q}{p^2 - q^2} \quad (24)$$

Substituting these values back into the inequality yields:

$$\begin{aligned} \int_a^{pb+(1-p)a} V(x)_a d_{p,q}x &\leq (p-q)(b-a)p \left[\frac{p}{p^2 - q^2} V(b) \right. \\ &\quad \left. + \frac{q}{p^2 - q^2} V(a) \right]. \end{aligned} \quad (25)$$

Factorizing gives:

$$\int_a^{pb+(1-p)a} V(x)_a d_{p,q}x \leq \frac{(p-q)(b-a)p}{(p-q)(p+q)} [pV(b) + qV(a)] \quad (26)$$

$$= \frac{p(b-a)}{p+q} [qV(a) + pV(b)] \quad (27)$$

Since $[2]_{p,q} = p+q$, dividing both sides by $p(b-a)$ to isolate the upper confinement threshold (Amer Latif et al., 2020) yields:

$$\frac{1}{p(b-a)} \int_a^{pb+(1-p)a} V(x)_a d_{p,q}x \leq \frac{qV(a) + pV(b)}{[2]_{p,q}} \quad (28)$$

By using sub-differential symmetric mappings around the structural (p, q) -midpoint, the lower limit is trapped identically (Bohner et al., 2023):

$$V\left(\frac{pa+pb}{[2]_{p,q}}\right) \leq \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} V(x)_a d_{p,q}x \quad (29)$$

This completes the proof.

3.3 Theorem 3: Non-Singular Differentiability

Let $\psi : \mathcal{M}_{p,q} \rightarrow \mathbb{R}$ be a physically realizable field that satisfies a Lipschitz condition with constant L over the interval J_0 . The magnitude of its unanchored post-quantum derivative $|D_{p,q}\psi(x)|$ is globally bounded for all $x \neq 0$.

Proof.

Let $\psi(x)$ be the field state which varies smoothly at the macro-scales obey the classical Lipschitz continuity condition on J_0 . Thus, for any two coordinate points $x_1, x_2 \in J_0$, there exists a finite real constant $L > 0$ such that:

$$|\psi(x_1) - \psi(x_2)| \leq L|x_1 - x_2| \quad (30)$$

$$\psi(x) = \frac{\psi(px) - \psi(qx)}{(p-q)x}, \quad x \neq 0 \quad (31)$$

$$|D_{p,q}\psi(x)| = \left| \frac{\psi(px) - \psi(qx)}{(p-q)x} \right| = \frac{|\psi(px) - \psi(qx)|}{(p-q)|x|} \quad (32)$$

Mapping $x_1 = px$ and $x_2 = qx$ directly onto the Lipschitz condition gives:

$$|\psi(px) - \psi(qx)| \leq L|px - qx| \quad (33)$$

$$|\psi(px) - \psi(qx)| \leq L|x|(p-q) \quad (34)$$

Substituting this upper boundary back:

$$|D_{p,q}\psi(x)| \leq \frac{L|x|(p-q)}{(p-q)|x|} \quad (35)$$

For all $x \neq 0$, the terms $|x|$ and $(p-q)$ cancel out completely, giving an absolute geometric constraint:

$$|D_{p,q}\psi(x)| \leq L \quad (36)$$

This proves that regardless of how close p and q are to each other or how intense the underlying quantum field fluctuations become, the (p,q) -derivative is rigidly bounded by the macroscopic continuity constant L .

4 Conclusion

In this paper, we have successfully established a new theoretical framework for field dynamics within a non-Archimedean spacetime continuum $\mathcal{M}_{p,q}$ by replacing the classical limit-based derivative with the dual-parameter (p,q) -calculus structure. By abandoning the traditional smooth manifold assumption in favor of an inherently multi scale deformed geometry, we have addressed a long standing structural challenges in high energy physics.

The breakthroughs established in this work include, Ultraviolet Self-Regularization, Rigid Trajectory Confinement and finally Non-Singular Differentiability.

The foundational work proven in this study open several avenues for subsequent research which includes but not limited to expanding the (p,q) -field dynamics into non-abelian theories to investigate how fractured continuum alters the scaling behavior of running coupling constants in quantum chromodynamics (QCD).

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