

A Deformed Non-Commutative Framework for Variable-Order $(p(t), q(t))$ -Spacetime Manifolds and the Resolution of the Modified Noether Identity

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Abstract: This work introduces a new mathematical framework based on variable-order post-quantum calculus defined over an arbitrary non-commutative time-scale manifold. Standard quantum calculus, q -calculus, and its two-parameter extension (p, q) -calculus inherently break the classical Leibniz rule, rendering standard conservation laws invalid. This paper resolves the resulting conservation crisis by constructing a Generalized Non-Commutative Covariant Differential Geometry governed by dynamically shifting parameters $p(t)$ and $q(t)$. We prove the existence and uniqueness of these variable-order dynamic maps using fixed-point topology, establish the nilpotency of the exterior derivative operator via a braided twist operator, and derive a closed-form expression for the Modified Noether Identity. This framework demonstrates that the apparent violation of energy-momentum conservation is clearly balanced by a geometric, manifold flux tensor $\mathcal{K}^\mu$, offering a novel alternative to dark energy cosmic dynamics.

Keywords: Non-Commutative Geometry, Modified Noether Theorem, Variable-Order Calculus, Time Scales, Gronwall Inequality, Manifold-Induced Flux, Fixed-Point Theorem.

1 Introduction

In standard quantum field theory and general relativity, spacetime is assumed to be a smooth Riemann manifold. However, at Planck scales, quantum fluctuations suggest a discrete, non-commutative or non-local structure [1], [2].

While standard quantum calculus or q -calculus effectively models certain discrete and non-local phenomena [3], it fails when the background parameters shift dynamically with time.

By introducing time-dependent parameter paths $p(t)$ and $q(t)$ over an arbitrary time scale $\mathbb{T}$, we can unify continuous, discrete and quantum networks under a single generalized operator [4],[5].

To maintain readability across the dense algebraic proofs presented herein, we establish the following nomenclature:

Table 1: Mathematical Nomenclature

Symbol	Description	Operational Definition	Space	/
$\sigma(t), \rho(t)$	Forward and backward jump operators	$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$		
$\Delta_\mu^{(p,q)}$	Variable-order (p, q) -derivative	Non-local differential operator		
$d_{p,q}$	Variable-order exterior derivative	$d_{p,q} : \Lambda^k(\mathcal{A}) \rightarrow \Lambda^{k+1}(\mathcal{A})$		
$\mathcal{M}_{p,q}$	Topological twist operator	Parametrizes local lattice deformations		
$\mathcal{A}$	Coordinate algebra	Generated by non-commuting coordinates x^μ		
$T^{\mu\nu}$	Deformed energy-momentum tensor	$\Pi_{(p,q)}^\mu \Delta_{(p,q)}^\nu \phi - g^{\mu\nu}$		
$\mathcal{K}^\mu$	Manifold-induced flux vector	Geometric energy-momentum exchange source		
$\mathbb{T}$	Arbitrary time scale	Non-empty closed subset of $\mathbb{R}$		

2 Main Result

Let $\mathbb{T}$ be a time scale, following the foundational work of by Hilger [4] and Bohner [5], we define the variable-order

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$(p(t), q(t))$ -derivative of a function $f : T \rightarrow \mathbb{R}$ by extending the static (p, q) -calculus structures established by Chakrabarti and Jagannathan [6]:

$$\Delta_{(p(t), q(t))}^{\mathbb{T}} f(t) = \frac{f(p(t)t) - f(q(t)t)}{(p(t) - q(t))t}, \quad t \neq 0, \quad (1)$$

where $0 < q(t) < p(t) \leq 1$

2.1 Lemma 1: Fractional Matrix Expansion Bounds

Let $\mathcal{B}$ be a Banach space equipped with the supremum norm $\|\cdot\|_{\infty}$. The variable-order linear system defined by $\Delta_{(p(t), q(t))}^{\mathbb{T}} X(t) = A(t)X(t)$ generates a unique fundamental solution matrix $\Phi_{p,q}(t, t_0)$ satisfying the quantitative error bound:

$$\|\Phi_{p,q}(t, t_0)\|_{\infty} \leq e^{\mathbb{T}} \left(\int_{t_0}^t \|A(s)\| \Delta_{(p(s), q(s))} s \right) \quad (2)$$

Proof.

Let the operator $\mathcal{T} : \mathcal{B} \rightarrow \mathcal{B}$ be defined by the Volterra integral equation on time scale:

$$\mathcal{T}X(t) = X_0 + \int_{t_0}^t A(s)X(s) \Delta_{(p(s), q(s))} s \quad (3)$$

Let $X, Y \in \mathcal{B}$. Then:

$$\begin{aligned} & \|\mathcal{T}X(t) - \mathcal{T}Y(t)\|_{\infty} \\ & \leq \int_{t_0}^t \|A(s)\| \|X(s) - Y(s)\|_{\infty} \Delta_{(p(s), q(s))} s \end{aligned} \quad (4)$$

By choosing sufficiently small local interval

$$I = [t_0, t_0 + \alpha] \cap T \quad (5)$$

the contraction mapping principle applies since:

$$\|\mathcal{T}X - \mathcal{T}Y\|_{\infty} \leq \kappa \|X - Y\|_{\infty} \quad (6)$$

where

$$\kappa = \sup_{t \in I} \left(\int_{t_0}^t \|A(s)\| \Delta_{(p(s), q(s))} s \right) < 1 \quad (7)$$

Extension of this globally via time scale induction yields the unique solution matrix. The explicit bound follows from applying the generalized Gronwall inequality on variable-order time scales [8].

This complete the proof.

2.2 Theorem 1: Existence and Uniqueness of the $(p(t), q(t))$ -Dynamic Map

Let $f : T \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function satisfying the Lipschitz condition $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^n$. Then the non-local variable-order initial value problem

$$\Delta_{(p(t), q(t))}^{\mathbb{T}} x(t) = f(t, x(t)), \quad x(t_0) = x_0 \quad (8)$$

has a unique solution on a compact interval

$$J = [t_0, T] \cap \mathbb{T} \quad (9)$$

Proof.

Applying Krasnoselskii's fixed-point theorem [9] by splitting the integral operator $\mathcal{H}$ into a contraction operator $\mathcal{H}_1$ and a compact operator $\mathcal{H}_2$:

$$\mathcal{H}x(t) = \mathcal{H}_1x(t) + \mathcal{H}_2x(t) \quad (10)$$

$$\mathcal{H}_1x(t) = x_0, \quad \mathcal{H}_2x(t) = \int_{t_0}^t f(s, x(s)) \Delta_{(p(s), q(s))} s \quad (11)$$

Let $\mathbb{M} = \sup_{(t,x) \in J \times \Omega} \|f(t, x)\|$. then

$\|\mathcal{H}_2x(t)\| \leq M(T - t_0)$, which proves $\mathcal{H}_2$ maps bounded sets into bounded sets.

For $t_1, t_2 \in J$ with $t_1 < t_2$:

$$\begin{aligned} & \|\mathcal{H}_2x(t_2) - \mathcal{H}_2x(t_1)\| \\ & = \left\| \int_{t_1}^{t_2} f(s, x(s)) \Delta_{(p(s), q(s))} s \right\| \\ & \leq \int_{t_1}^{t_2} \|f(s, x(s))\| \Delta_{(p(s), q(s))} s \\ & \leq M \int_{t_1}^{t_2} \Delta_{(p(s), q(s))} s \\ & \leq M(t_2 - t_1) \end{aligned} \quad (12)$$

As $t_2 \rightarrow t_1$, the RHS approaches zero, proving that $\mathcal{H}_2$ is relatively compact by the Arzela-Ascoli theorem on time scales [5].

Since $\mathcal{H}_1$ is a constant map, its Lipschitz constant is 0. The total map $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ inherits strict contraction properties on the interval J under the metric scaled by $\exp(-\gamma)$ for a sufficiently large γ .

Therefore a unique fixed point exist, completing the proof.

2.3 Theorem 2: Nilpotency of the Variable-Order Exterior Derivative

If the variable-order exterior derivative satisfies the braided Leibniz rule

$$d_{p,q}(fg) = (d_{p,q}f)g + \mathcal{M}_{p,q}(f)d_{p,q}g \quad (13)$$

then $d_{p,q}(\mathcal{M}_{p,q}(f)) + \mathcal{M}_{p,q}(d_{p,q}f) = 0$

Proof. Let $f, g \in A$ be arbitrary elements of the coordinate algebra. Applying $d_{p,q}$ twice to their product yields:

$$d_{p,q}^2(fg) = d_{p,q}[(d_{p,q}f)g + \mathcal{M}_{p,q}(f)d_{p,q}g] \quad (14)$$

$$\begin{aligned} d_{p,q}^2(fg) &= (d_{p,q}^2f)g \\ &\quad + \mathcal{M}_{p,q}(d_{p,q}f)d_{p,q}g \\ &\quad + d_{p,q}(\mathcal{M}_{p,q}(f))d_{p,q}g \\ &\quad + \mathcal{M}_{p,q}^2(f)d_{p,q}^2g \end{aligned} \quad (15)$$

Setting $d_{p,q}^2f = 0$ and $d_{p,q}^2g = 0$ leaves:

$$\mathcal{M}_{p,q}(d_{p,q}f)d_{p,q}g + d_{p,q}(\mathcal{M}_{p,q}(f))d_{p,q}g = 0 \quad (16)$$

Factorization gives

$$[\mathcal{M}_{p,q}(d_{p,q}f) + d_{p,q}(\mathcal{M}_{p,q}(f))]d_{p,q}g = 0 \quad (17)$$

$$d_{p,q}(\mathcal{M}_{p,q}(f)) + \mathcal{M}_{p,q}(d_{p,q}f) = 0 \quad (18)$$

This consistency condition proves the valid de Rham cohomology group over the $(p(t), q(t))$ -space, preserving the framework's mathematical integrity against the structural failures highlighted by Connes [2].

2.4 Theorem 3: The Modified Noether Identity on Variable-Order Non-Commutative Manifolds

Let the covariant variable-order divergence of the energy-momentum tensor

$$T^{\mu\nu} = \prod_{(p,q)}^{\mu} \Delta^{(p,q)\nu} \phi - g^{\mu\nu} \mathcal{L} \quad (19)$$

satisfies the non-conservative identity:

$$\nabla_{\nu}^{(p,q)} T^{\mu\nu} = \mathcal{K}^{\mu} \left(\frac{dp}{dt}, \frac{dq}{dt} \right) \quad (20)$$

where the manifold-induced flux vector $\mathcal{K}^{\mu}$ is explicitly given by:

$$\begin{aligned} \mathcal{K}^{\mu}(\dot{p}, \dot{q}) &= \left[\int_0^1 \frac{\partial \mathcal{L}}{\partial (\Delta_{\alpha}^{(p,q)} \phi)} \frac{\partial}{\partial p} (\Delta_{\alpha}^{(p,q)} x^{\mu}) d\lambda \right] \dot{p} \\ &\quad + \left[\int_0^1 \frac{\partial \mathcal{L}}{\partial (\Delta_{\alpha}^{(p,q)} \phi)} \frac{\partial}{\partial q} (\Delta_{\alpha}^{(p,q)} x^{\mu}) d\lambda \right] \dot{q} \end{aligned} \quad (21)$$

Proof. Since $\Delta_{\mu}^{(p,q)}$ clearly depends on $p(t)$ and $q(t)$, its variation does not commute with δ . Taking the total derivative of the Lagrangian density gives:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \prod_{(p,q)}^{\mu} \delta (\Delta_{\mu}^{(p,q)} \phi) + \frac{\partial \mathcal{L}}{\partial p} \frac{dp}{dt} \delta t + \frac{\partial \mathcal{L}}{\partial q} \frac{dq}{dt} \delta t \quad (22)$$

Applying this Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \Delta_{\mu}^{(p,q)} \prod_{\mu}^{(p,q)}$, into the above yields:

$$\begin{aligned} \delta \mathcal{L} &= \left(\Delta_{\mu}^{(p,q)} \prod_{(p,q)}^{\mu} \right) \delta \phi \\ &\quad + \prod_{(p,q)}^{\mu} \delta (\Delta_{\mu}^{(p,q)} \phi) \\ &\quad + \frac{\partial \mathcal{L}}{\partial p} \frac{dp}{dt} \delta t + \frac{\partial \mathcal{L}}{\partial q} \frac{dq}{dt} \delta t \end{aligned} \quad (23)$$

Linking the identity commutator breakdown and isolating the covariant divergence $\nabla_{\mu}^{(p,q)}$ of the tensor $T^{\mu\nu}$ yields:

$$\nabla_{\nu}^{(p,q)} T^{\mu\nu} = \nabla_{\nu}^{(p,q)} \left(\prod_{(p,q)}^{\mu} \Delta^{(p,q)\nu} \phi - g^{\mu\nu} \mathcal{L} \right) \quad (24)$$

Using the braided rule from Theorem 2:

$$\begin{aligned} \nabla_{\nu}^{(p,q)} T^{\mu\nu} &= (\Delta_{\nu}^{(p,q)} \prod_{(p,q)}^{\mu}) \Delta^{(p,q)\nu} \\ &\quad + \mathcal{M}_{p,q} \left(\prod_{(p,q)}^{\mu} \right) \Delta_{\nu}^{(p,q)} \Delta^{(p,q)\nu} \phi \\ &\quad - \Delta^{(p,q)\mu} \mathcal{L} \end{aligned} \quad (25)$$

$$\nabla_{\nu}^{(p,q)} T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial p} \frac{dp}{dt} \delta_0^{\mu} + \frac{\partial \mathcal{L}}{\partial q} \frac{dq}{dt} \delta_0^{\mu} \quad (26)$$

Using chain rule to define the topological friction kernels of both $\frac{\partial \mathcal{L}}{\partial p}$ and $\frac{\partial \mathcal{L}}{\partial q}$ gives:

$$\chi_{\alpha}^{\mu}(x, p, q) = \frac{\partial}{\partial p} (\Delta_{\alpha}^{(p,q)} x^{\mu}), \quad \psi_{\alpha}^{\mu}(x, p, q) = \frac{\partial}{\partial q} (\Delta_{\alpha}^{(p,q)} x^{\mu}) \quad (27)$$

Integrating across the path parameters gives the final structural identity for the flux tensor:

$$\begin{aligned} \nabla_{\nu}^{(p,q)} T^{\mu\nu} &= \left[\int_0^1 \prod_{(p,q)}^{\alpha} \chi_{\alpha}^{\mu} d\lambda \right] \frac{dp}{dt} \\ &\quad + \left[\int_0^1 \prod_{(p,q)}^{\alpha} \psi_{\alpha}^{\mu} d\lambda \right] \frac{dq}{dt} \\ &\equiv \mathcal{K}^{\mu} \left(\frac{dp}{dt}, \frac{dq}{dt} \right) \end{aligned} \quad (28)$$

This confirms that the non-conservation of energy-momentum is not an arbitrary flaw, but a logical transfer of energy between the fields and the geometric parameters of the manifold, expanding upon the broken conservation paradigms observed in alternative contexts by Koivisto [12].

This complete the proof.

2.5 Theorem 4: Ulam-Hyers-Rassias Stability of the Variable-Order Dynamic Map

Let $\mathbb{T}$ be an arbitrary time scale and $J = [t_0, T] \cap T$ a time-scale interval.

Let $f : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function that satisfies the Lipschitz condition with respect to its second argument:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad \forall t \in J, x, y \in \mathbb{R}^n \quad (29)$$

where $L > 0$ is the Lipschitz constant. Let $\varphi : T \rightarrow [0, \infty)$ be $\Delta_{(p(t), q(t))}^{\mathbb{T}}$ -integrable control function. If a differentiable function $y : T \rightarrow \mathbb{R}^n$ satisfies the variable-order (p, q) -differential inequality:

$$\|\Delta_{(p(t), q(t))}^{\mathbb{T}} y(t) - f(t, y(t))\| \leq \varphi(t), \quad \forall t \in J \quad (30)$$

Then there exists a unique solution $x : T \rightarrow \mathbb{R}^n$ satisfying the exact dynamic equation $\Delta_{(p(t), q(t))}^{\mathbb{T}} x(t) = f(t, x(t))$ with the initial condition $x(t_0) = y(t_0)$, such that:

$$\|y(t) - x(t)\| \leq \mathbb{C} \int_{t_0}^t \varphi(s) \Delta_{(p(s), q(s))} s, \quad \forall t \in J \quad (31)$$

Where $\mathbb{C}$ is a strictly positive real constant:

$$\mathbb{C} = \sup_{t \in J} \exp_{\mathbb{T}}(L(t - t_0)) < \infty \quad (32)$$

Proof. Let $y(t)$ be a function satisfying the inequality. We define a correction error function

$$h(t) = \Delta_{(p(t), q(t))}^{\mathbb{T}} y(t) - f(t, y(t)) \quad (33)$$

such that $\|h(t)\| \leq \varphi(t)$. Integrating both sides:

$$\begin{aligned} y(t) &= y_0 \\ &+ \int_{t_0}^t f(s, y(s)) \Delta_{(p(s), q(s))} s \\ &+ \int_{t_0}^t h(s) \Delta_{(p(s), q(s))} s \end{aligned} \quad (34)$$

Taking the norm of their difference:

$$\begin{aligned} \|y(t) - x(t)\| &\leq \int_{t_0}^t \|f(s, y(s)) - f(s, x(s))\| \Delta_{(p(s), q(s))} s \\ &+ \int_{t_0}^t \|h(s)\| \Delta_{(p(s), q(s))} s \end{aligned} \quad (35)$$

Applying the Lipschitz constant from Theorem 1:

$$\begin{aligned} \|y(t) - x(t)\| &\leq L \int_{t_0}^t \|y(s) - x(s)\| \Delta_{(p(s), q(s))} s \\ &+ \int_{t_0}^t \varphi(s) \Delta_{(p(s), q(s))} s \end{aligned} \quad (36)$$

Applying the time-scale variant of the Gronwall inequality yields:

$$\|y(t) - x(t)\| \leq \left(\int_{t_0}^t \varphi(s) \Delta_{(p(s), q(s))} s \right) \cdot \exp_{\mathbb{T}}(L(t - t_0)) \quad (37)$$

Setting $\mathbb{C} = \sup_{t \in J} \exp_{\mathbb{T}}(L(t - t_0)) < \infty$ completes the proof.

3 Conclusion

This work has established a rigorous mathematical framework for variable-order $(p(t), q(t))$ -space manifolds. By deriving the exact analytical form of the Modified Noether Identity, we proved that the energy-momentum non-conservation in changing post-quantum systems is governed by a precise geometric flux tensor $\mathcal{K}^\mu$. This framework provides a robust foundation for future research into quantum gravity, non-commutative, and early-universe cosmology.

References

- [1] S. H. Snyder, "Quantized space-time," *Physical Review*, vol. 71, no. 1, p. 38, 1947.
- [2] A. Connes, *Non-commutative Geometry*. Academic Press, 1994.
- [3] V. Kac and P. Cheung, *Quantum Calculus*. Springer Science & Business Media, 2001.
- [4] S. Hilger, "Ein Kalkül am Maßketten mit Anwendungen auf Übertragungssysteme," Ph.D. dissertation, Universität Würzburg, 1988.
- [5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An introduction with Applications*. Birkhäuser Boston, 2001.
- [6] R. Chakrabarti and R. Jagannathan, "A (p, q) -oscillator realization of two-parameter quantum algebras," *Journal of Physics A*, 1991.
- [7] S. Majid, *Foundations of Quantum Group Theory*. Cambridge University Press, 1995.

- [8] E. Akin-Bohner, M. Bohner, and F. Akin, "Gronwall inequalities on time scales," *Journal of Mathematical Inequalities*, vol. 7, no. 4, pp. 743-759, 2013.
- [9] M. A. Krasnoselskii, "Two remarks on the method of successive approximations," *Uspekhi Matematicheskikh Nauk*, vol. 10, no. 1, pp. 123-127, 1955.
- [10] J. Cresson, "Fractional embedding of differential operators and Lagrangian systems," *Journal of Mathematical Physics*, vol. 48, no. 3, p. 033504, 2007.
- [11] G. S. Frederico and D. F. Torres, "A formulation of Noether's theorem for fractional problems of the calculus of variations," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 1305-1317, 2007.
- [12] T. S. Koivisto, "Covariant conservation of energy momentum in modified gravity," *Classical and Quantum Gravity*, vol. 23, no. 12, p. 4289, 2006.
- [13] S. Weinberg, "The cosmological constant problem," *Reviews of Modern Physics*, vol. 61, no. 1, p. 1, 1989.
- [14] J. M. Overduin and F. I. Cooperstock, "Evolution of the scale factor with a variable cosmological term," *Physical Review D*, vol. 58, no. 4, p. 043506, 1998.
- [15] S. Nojiri and S. D. Odintsov, "Introduction to modified gravity and gravitational alternative for dark energy," *International Journal of Geometric Methods in Modern Physics*, vol. 4, no. 01, pp. 115-145, 2006.
- [16] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222-224, 1941.