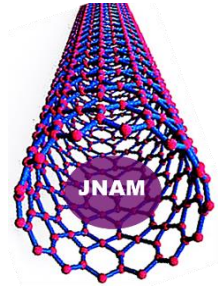


On Estimation Residual Stresses in Metals After Heat Treatment

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Abstract: In this paper we consider an approach for estimation a residual stress in metals after heat treatment. In the framework of the considered approach, we introduce a model of changing of stress in metals during heat treatment as well as analytical approach for its analysis are proposed.

Keywords: heat treatment of metals; residual stresses in metals; analytical approach for analysis.

1. Introduction

Heat treatment is one of the main stages of the production of metal devices. Quality and resource of these devices depend on its correct implementation. The main factors that ensure the quality of heat treatment are temperature and time. Thus, the above properties of these devices are determined by the cooling rate of the metal [1-4]. There are different approaches, that gives a possibility to control the cooling rate of metal products, were elaborated. The manufacturing of different structures in a metallic material is accompanied by the redistribution of various structural defects (atoms, dislocations, ...) [1,2,4,5]. The final fixed structural state of the metal is characterized by an internal stress consisting of the interaction of elastic fields of lattice stresses and defects. Residual stresses arising during heat treatment, combined with workloads, have a negative impact on operational reliability and service life. The main aim of the present paper is formulation and analysis of a model of changes of stress in metals during heat treatment. The accompanying aim of the present paper is to elaboration of an analytical approach to analyze the considered model.

Method of solution

To solve the considered aim we determine distribution of temperature $T(x, y, z, t)$ in space and time by solving of the second Fourier's law [6,7]

$$c(T) \frac{\partial T(x, y, z, t)}{\partial t} = \frac{\partial}{\partial x} \left[\lambda(x, y, z, T) \frac{\partial T(x, y, z, t)}{\partial x} \right] + \frac{\partial}{\partial y} \left[\lambda(x, y, z, T) \frac{\partial T(x, y, z, t)}{\partial y} \right] + \frac{\partial}{\partial z} \left[\lambda(x, y, z, T) \frac{\partial T(x, y, z, t)}{\partial z} \right] + p(x, y, z, t) \quad (1)$$

with boundary and initial conditions

$$\begin{aligned} \left. \frac{\partial T(x, y, z, t)}{\partial x} \right|_{x=0} &= 0, \quad \left. \frac{\partial T(x, y, z, t)}{\partial x} \right|_{x=L_x} = 0; \\ \left. \frac{\partial T(x, y, z, t)}{\partial y} \right|_{y=0} &= 0, \quad \left. \frac{\partial T(x, y, z, t)}{\partial y} \right|_{y=L_y} = 0; \\ \left. \frac{\partial T(x, y, z, t)}{\partial z} \right|_{z=0} &= 0, \quad \left. \frac{\partial T(x, y, z, t)}{\partial z} \right|_{z=L_z} = 0; \\ T(x, y, z, 0) &= T_r. \end{aligned} \quad (2)$$

Here λ is the coefficient of thermal conductivity. Value of the coefficient depends on properties of the considered materials and temperature. Dependence of the coefficient of thermal conductivity could be approximated by the following function: $\lambda(x, y, z, T) = \lambda_{ass}(x, y, z) [1 + \mu T_d^q / T^q(x, y, z, t)]$. $c(T) = c_{ass} [1 - \theta \exp(-T(x, y, z, t) / T_d)]$ is the heat capacitance; T_d is the Debye temperature. In the case, when current temperature $T(x, y, z, t)$ is equal or larger, then the Debye temperature T_d , then we can use the following approximation $c(T) \approx c_{ass}$. This is most interest for us case. $p(x, y, z, t)$ is the volumetric density of power, which released in the considered sample. First of all we estimate distribution of temperature. To make the estimation we solve the equation (1) by the method of averaging of function corrections [8-10]. In the framework of this method, we replace the distribution of temperature in space and time by not yet known average value α_{1T} in the right side of equation (1). The replacement gives a possibility to obtain the following equation to determine the first-order approximation of the considered temperature

$$c_{ass} \frac{\partial T_1(x, y, z, t)}{\partial t} = p(x, y, z, t) \quad (3)$$

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Integration of left and right sides of equation (3) on time gives a possibility to obtain the required approximation in the following form

$$T_1(x, y, z, t) = \frac{1}{c_{ass}} \int_0^t p(x, y, z, \tau) d\tau + T_r \quad (4)$$

Not yet known average value α_{1T} was calculated by the following standard relation [8-10]

$$\alpha_{1T} = \frac{1}{\Theta L_x L_y L_z} \int_0^{\Theta L_x} \int_0^{\Theta L_y} \int_0^{\Theta L_z} T_1(x, y, z, t) dz dy dx dt \quad (5)$$

The second-order approximation of temperature was calculated by standard replacing of the required function $T(x, y, z, t)$ on sum of average value of the considered approximation and approximation with the previous order (i.e. $T(x, y, z, t) \rightarrow \alpha_{1T} + T_1(x, y, z, t)$) in the right side of equation (1). The replacement gives a possibility to obtain equation to calculate the considered second-order approximation in the following form

$$\begin{aligned} c_{ass} \frac{\partial T_2(x, y, z, t)}{\partial t} = & \frac{\partial}{\partial x} \left(\lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial x} \right) + \\ & + \frac{\partial}{\partial y} \left(\lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial y} \right) + \\ & + \frac{\partial}{\partial z} \left(\lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial z} \right) + p(x, y, z, t) \end{aligned} \quad (6)$$

Termwise differentiation gives a possibility to transform the equation to the following form

$$\begin{aligned} c_{ass} \frac{\partial T_2(x, y, z, t)}{\partial t} = & \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial x} \frac{\partial \lambda_{ass}(x, y, z)}{\partial x} - \lambda_{ass}(x, y, z) \times \\ & \times \varphi \mu T_d^\varphi \left[\frac{\partial T_1(x, y, z, t)}{\partial x} \right]^2 \frac{1}{[\alpha_{2T} + T_1(x, y, z, t)]^{\varphi+1}} + \lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \times \\ & \times \frac{\partial^2 T_1(x, y, z, t)}{\partial x^2} + \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial y} \frac{\partial \lambda_{ass}(x, y, z)}{\partial y} - \left[\frac{\partial T_1(x, y, z, t)}{\partial y} \right]^2 \times \\ & \times \frac{\varphi \mu T_d^\varphi \lambda_{ass}(x, y, z)}{[\alpha_{2T} + T_1(x, y, z, t)]^{\varphi+1}} + \lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial^2 T_1(x, y, z, t)}{\partial y^2} + \\ & + \frac{\partial T_1(x, y, z, t)}{\partial z} \frac{\partial \lambda_{ass}(x, y, z)}{\partial z} \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} - \frac{\varphi \mu T_d^\varphi \lambda_{ass}(x, y, z)}{[\alpha_{2T} + T_1(x, y, z, t)]^{\varphi+1}} \times \\ & \times \left[\frac{\partial T_1(x, y, z, t)}{\partial z} \right]^2 + \lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial^2 T_1(x, y, z, t)}{\partial z^2} + p(x, y, z, t) \end{aligned} \quad (6a)$$

Integration of the above equation on time gives a possibility to obtain equation to determine the second-order approximation of temperature in the following form

$$\begin{aligned}
c_{ass} T_2(x, y, z, t) = & \frac{\partial \lambda_{ass}(x, y, z)}{\partial x} \int_0^t \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, \tau)]^\varphi} \right\} \frac{\partial T_1(x, y, z, \tau)}{\partial x} d\tau - \\
& - \varphi \mu T_d^\varphi \int_0^t \left[\frac{\partial T_1(x, y, z, \tau)}{\partial x} \right]^2 \frac{\lambda_{ass}(x, y, z) d\tau}{[\alpha_{2T} + T_1(x, y, z, \tau)]^{\varphi+1}} + \lambda_{ass}(x, y, z) \int_0^t \frac{\partial^2 T_1(x, y, z, \tau)}{\partial x^2} \times \\
& \times \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, \tau)]^\varphi} \right\} d\tau + \int_0^t \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, \tau)]^\varphi} \right\} \frac{\partial T_1(x, y, z, \tau)}{\partial y} d\tau \times \\
& \times \frac{\partial \lambda_{ass}(x, y, z)}{\partial y} - \int_0^t \left[\frac{\partial T_1(x, y, z, \tau)}{\partial y} \right]^2 \frac{\varphi \mu T_d^\varphi \lambda_{ass}(x, y, z)}{[\alpha_{2T} + T_1(x, y, z, \tau)]^{\varphi+1}} d\tau + \lambda_{ass}(x, y, z) \times \\
& \times \int_0^t \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, \tau)]^\varphi} \right\} \frac{\partial^2 T_1(x, y, z, \tau)}{\partial y^2} d\tau + \int_0^t \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, \tau)]^\varphi} \right\} \times \\
& \times \frac{\partial T_1(x, y, z, \tau)}{\partial z} d\tau \frac{\partial \lambda_{ass}(x, y, z)}{\partial z} - \varphi \mu \int_0^t \left[\frac{\partial T_1(x, y, z, \tau)}{\partial z} \right]^2 \frac{1}{[\alpha_{2T} + T_1(x, y, z, \tau)]^{\varphi+1}} d\tau \times \\
& \times T_d^\varphi \lambda_{ass}(x, y, z) + \int_0^t p(x, y, z, \tau) d\tau + \int_0^t \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, \tau)]^\varphi} \right\} \frac{\partial^2 T_1(x, y, z, \tau)}{\partial z^2} d\tau \times \\
& \times \lambda_{ass}(x, y, z).
\end{aligned} \tag{6b}$$

Average value α_{T2} was calculated by using the following standard relation [8-10]

$$\alpha_{2T} = \frac{1}{\Theta L_x L_y L_z} \int_0^\Theta \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} [T_2(x, y, z, t) - T_1(x, y, z, t)] dz dy dx dt \tag{7}$$

Substitution of the first- and the second-order approximations of the considered temperature gives a possibility to obtain the following equation to determine the average value α_{T2}

$$\begin{aligned}
& \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \frac{\partial \lambda_{ass}(x, y, z)}{\partial x} \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial x} dz dy dx dt - \varphi \mu \times \\
& \times T_d^\varphi \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[\frac{\partial T_1(x, y, z, t)}{\partial x} \right]^2 \frac{\lambda_{ass}(x, y, z) dz dy dx dt}{[\alpha_{2T} + T_1(x, y, z, t)]^{\varphi+1}} + \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \lambda_{ass}(x, y, z) \times \\
& \times \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial^2 T_1(x, y, z, t)}{\partial x^2} dz dy dx dt + \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \frac{\partial \lambda_{ass}(x, y, z)}{\partial y} \times \\
& \times \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial y} dz dy dx dt - \varphi \mu T_d^\varphi \int_0^\Theta \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[\frac{\partial T_1(x, y, z, t)}{\partial y} \right]^2 \times
\end{aligned}$$

$$\begin{aligned}
& \times \frac{\lambda_{ass}(x, y, z) dz dy dx}{[\alpha_{2T} + T_1(x, y, z, \tau)]^{\varphi+1}} (\Theta - t) dt + \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \lambda_{ass}(x, y, z) \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \times \\
& \times \frac{\partial^2 T_1(x, y, z, t)}{\partial y^2} dz dy dx dt + \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \frac{\partial T_1(x, y, z, t)}{\partial z} \times \\
& \times \frac{\partial \lambda_{ass}(x, y, z)}{\partial z} dz dy dx dt - \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[\frac{\partial T_1(x, y, z, t)}{\partial z} \right]^2 \frac{\lambda_{ass}(x, y, z) dz dy dx}{[\alpha_{2T} + T_1(x, y, z, t)]^{\varphi+1}} dt \times \\
& \times \varphi \mu T_d^\varphi + \int_0^\Theta (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} p(x, y, z, t) dz dy dx dt + \int_0^\Theta \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left\{ 1 + \frac{\mu T_d^\varphi}{[\alpha_{2T} + T_1(x, y, z, t)]^\varphi} \right\} \times \\
& \times \lambda_{ass}(x, y, z) \frac{\partial^2 T_1(x, y, z, t)}{\partial z^2} dz dy dx (\Theta - t) dt = 0
\end{aligned} \tag{8}$$

Solution of the above equation (i.e. average value α_{T2}) depends on value of parameter φ . Solution of the above equation could be obtained by standard approaches [11]. Stress in metals was estimated by solution of the following system of equations [12]

$$\begin{cases}
\rho(x, y, z, t) \frac{\partial^2 u_x(x, y, z, t)}{\partial t^2} = \frac{\partial \sigma_{xx}(x, y, z, t)}{\partial x} + \frac{\partial \sigma_{xy}(x, y, z, t)}{\partial y} + \frac{\partial \sigma_{xz}(x, y, z, t)}{\partial z} \\
\rho(x, y, z, t) \frac{\partial^2 u_y(x, y, z, t)}{\partial t^2} = \frac{\partial \sigma_{yx}(x, y, z, t)}{\partial x} + \frac{\partial \sigma_{yy}(x, y, z, t)}{\partial y} + \frac{\partial \sigma_{yz}(x, y, z, t)}{\partial z} \\
\rho(x, y, z, t) \frac{\partial^2 u_z(x, y, z, t)}{\partial t^2} = \frac{\partial \sigma_{zx}(x, y, z, t)}{\partial x} + \frac{\partial \sigma_{zy}(x, y, z, t)}{\partial y} + \frac{\partial \sigma_{zz}(x, y, z, t)}{\partial z}
\end{cases} \tag{9}$$

$$\sigma_{ij} = K \frac{\partial u_k(x, y, z, t)}{\partial x_k} \delta_{ij} + \frac{E}{2(1+\gamma)} \times$$

Here σ_{ij} is the stress tensor, which equal to

$$\times \left[\frac{\partial u_i(x, y, z, t)}{\partial x_j} + \frac{\partial u_j(x, y, z, t)}{\partial x_i} - \frac{\delta_{ij}}{3} \frac{\partial u_k(x, y, z, t)}{\partial x_k} \right] - K \beta [T(x, y, z, t) - T_r]$$

ρ is the density of metal; δ_{ij} is the Kronecker symbol; E is the tensile modulus (Young modulus); u_i, u_j are the components $u_x(x, y, z, t)$, $u_y(x, y, z, t)$ and $u_z(x, y, z, t)$ of the displacement vector $\vec{u}(x, y, z, t)$; x_i, x_j are the coordinate x, y, z ; γ is Poisson coefficient; $\varepsilon_0 = (a_s - a_{EL})/a_{EL}$ is the mismatch parameter; a_s, a_{EL} are lattice distances in different areas of the considered

material; K is the modulus of uniform compression; β is the coefficient of thermal expansion; T_r is the equilibrium temperature, which coincide (for our case) with room temperature. With account of the above relation for the stress tensor the system of equation (9) could be written as

$$\rho \frac{\partial^2 u_x(x, y, z, t)}{\partial t^2} = \left[K + \frac{5E}{6(1+\gamma)} \right] \frac{\partial^2 u_x(x, y, z, t)}{\partial x^2} + \left[K - \frac{E}{3(1+\gamma)} \right] \frac{\partial^2 u_y(x, y, z, t)}{\partial x \partial y} +$$

$$\begin{aligned}
& + \frac{E}{2(1+\gamma)} \left[\frac{\partial^2 u_y(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u_z(x, y, z, t)}{\partial z^2} \right] + \left[K + \frac{E}{3(1+\gamma)} \right] \frac{\partial^2 u_z(x, y, z, t)}{\partial x \partial z} - \\
& - \beta K \frac{\partial T(x, y, z, t)}{\partial x} \\
\rho \frac{\partial^2 u_y(x, y, z, t)}{\partial t^2} & = \frac{E}{2(1+\gamma)} \left[\frac{\partial^2 u_y(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u_x(x, y, z, t)}{\partial x \partial y} \right] - K \beta \frac{\partial T(x, y, z, t)}{\partial y} + \\
& + \left[\frac{5E}{12(1+\gamma)} + K \right] \frac{\partial^2 u_y(x, y, z, t)}{\partial y^2} + \frac{\partial}{\partial z} \left\{ \frac{E}{2(1+\gamma)} \left[\frac{\partial u_y(x, y, z, t)}{\partial z} + \frac{\partial u_z(x, y, z, t)}{\partial y} \right] \right\} + \\
& + \left[K - \frac{E}{6(1+\gamma)} \right] \frac{\partial^2 u_y(x, y, z, t)}{\partial y \partial z} + K \frac{\partial^2 u_y(x, y, z, t)}{\partial x \partial y} \\
\rho \frac{\partial^2 u_z(x, y, z, t)}{\partial t^2} & = \left[\frac{\partial^2 u_z(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u_z(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u_x(x, y, z, t)}{\partial x \partial z} + \frac{\partial^2 u_y(x, y, z, t)}{\partial y \partial z} \right] \times \\
& \times \frac{E}{2(1+\sigma)} + \frac{\partial}{\partial z} \left\{ K \left[\frac{\partial u_x(x, y, z, t)}{\partial x} + \frac{\partial u_y(x, y, z, t)}{\partial y} + \frac{\partial u_z(x, y, z, t)}{\partial z} \right] \right\} + \frac{1}{6} \frac{\partial}{\partial z} \left\{ \frac{E}{1+\sigma} \times \right. \\
& \times \left[6 \frac{\partial u_z(x, y, z, t)}{\partial z} - \frac{\partial u_x(x, y, z, t)}{\partial x} - \frac{\partial u_y(x, y, z, t)}{\partial y} - \frac{\partial u_z(x, y, z, t)}{\partial z} \right] \right\} - K \beta \frac{\partial T(x, y, z, t)}{\partial z} .
\end{aligned} \tag{10}$$

Conditions for the above equations could be written as

$$\begin{aligned}
\frac{\partial \bar{u}(0, y, z, t)}{\partial x} &= 0 \quad ; \quad \frac{\partial \bar{u}(L_x, y, z, t)}{\partial x} = 0 \quad ; \quad \frac{\partial \bar{u}(x, 0, z, t)}{\partial y} = 0 \quad ; \quad \frac{\partial \bar{u}(x, L_y, z, t)}{\partial y} = 0 \quad ; \\
\frac{\partial \bar{u}(x, y, 0, t)}{\partial z} &= 0 \quad ; \quad \frac{\partial \bar{u}(x, y, L_z, t)}{\partial z} = 0 \quad ; \quad \bar{u}(x, y, z, 0) = \bar{u}_0 \quad ; \quad \bar{u}(x, y, z, \infty) = \bar{u}_0 .
\end{aligned}$$

Now we determine solutions of equations (10). To determine the first-order approximations of the considered components in the framework of the method of averaging of functions corrections [8-10] we substitute not yet known average values

α_i of the considered components instead of the required functions. The substitution gives a possibility to obtain the following equations to determine the considered approximations

$$\begin{aligned}
\rho \frac{\partial^2 u_{1x}(x, y, z, t)}{\partial t^2} &= -K \beta \frac{\partial T(x, y, z, t)}{\partial x} \quad , \quad \rho \frac{\partial^2 u_{1y}(x, y, z, t)}{\partial t^2} = -K \beta \frac{\partial T(x, y, z, t)}{\partial y} \quad , \\
\rho \frac{\partial^2 u_{1z}(x, y, z, t)}{\partial t^2} &= -K \beta \frac{\partial T(x, y, z, t)}{\partial z} .
\end{aligned}$$

Integration of left and right sides of the above equations on time t leads to the following results

$$u_{1x}(x, y, z, t) = \int_0^t \int_0^g \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial x} d\tau d\vartheta - \int_0^\infty \int_0^g \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial x} d\tau d\vartheta + u_{0x},$$

$$u_{1y}(x, y, z, t) = \int_0^t \int_0^g \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial y} d\tau d\vartheta - \int_0^\infty \int_0^g \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial y} d\tau d\vartheta + u_{0y},$$

$$u_{1z}(x, y, z, t) = \int_0^t \int_0^g \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial z} d\tau d\vartheta - \int_0^\infty \int_0^g \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial z} d\tau d\vartheta + u_{0z}$$

Approximations of the considered components of the displacement vector with the second order and approximations with higher order could be calculated by standard replacement of the required functions in equations (8) on the following sum $\alpha_i + u_{i-1}(x, y, z, t)$ [8-10]. The replacement gives a possibility to obtain equations to calculate the second-order approximations in the following form

$$\begin{aligned} \rho \frac{\partial^2 u_{2x}(x, y, z, t)}{\partial t^2} = & \left[K + \frac{5E}{6(1+\gamma)} \right] \frac{\partial^2 u_{1x}(x, y, z, t)}{\partial x^2} + \left[K(z) - \frac{E}{3(1+\gamma)} \right] \frac{\partial^2 u_{1y}(x, y, z, t)}{\partial x \partial y} + \\ & + \left[K + \frac{E}{3(1+\gamma)} \right] \frac{\partial^2 u_{1z}(x, y, z, t)}{\partial x \partial z} + \frac{E}{2(1+\gamma)} \left[\frac{\partial^2 u_{1y}(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u_{1z}(x, y, z, t)}{\partial z^2} \right] - \\ & - K \beta \frac{\partial T(x, y, z, t)}{\partial x} \end{aligned}$$

$$\begin{aligned} \rho \frac{\partial^2 u_{2y}(x, y, z, t)}{\partial t^2} = & \frac{E}{2(1+\gamma)} \left[\frac{\partial^2 u_{1y}(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u_{1x}(x, y, z, t)}{\partial x \partial y} \right] + \left[K + \frac{5E}{12(1+\gamma)} \right] \times \\ & \times \frac{\partial^2 u_{1y}(x, y, z, t)}{\partial y^2} + \frac{\partial}{\partial z} \left\{ \frac{E}{2(1+\gamma)} \left[\frac{\partial u_{1y}(x, y, z, t)}{\partial z} + \frac{\partial u_{1z}(x, y, z, t)}{\partial y} \right] \right\} + \left[K - \frac{E}{6(1+\gamma)} \right] \times \\ & \times \frac{\partial^2 u_{1y}(x, y, z, t)}{\partial y \partial z} + K \frac{\partial^2 u_{1y}(x, y, z, t)}{\partial x \partial y} - K \beta \frac{\partial T(x, y, z, t)}{\partial y} \end{aligned}$$

$$\begin{aligned} \rho \frac{\partial^2 u_{2z}(x, y, z, t)}{\partial t^2} = & \frac{E}{2(1+\gamma)} \left[\frac{\partial^2 u_{1z}(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u_{1x}(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u_{1x}(x, y, z, t)}{\partial x \partial z} + \right. \\ & + \left. \frac{\partial^2 u_{1y}(x, y, z, t)}{\partial y \partial z} \right] + \frac{\partial}{\partial z} \left\{ K(z) \left[\frac{\partial u_{1x}(x, y, z, t)}{\partial x} + \frac{\partial u_{1y}(x, y, z, t)}{\partial y} + \frac{\partial u_{1x}(x, y, z, t)}{\partial z} \right] \right\} + \\ & + \frac{\partial}{\partial z} \left\{ \frac{E}{6(1+\gamma)} \left[6 \frac{\partial u_{1z}(x, y, z, t)}{\partial z} - \frac{\partial u_{1x}(x, y, z, t)}{\partial x} - \frac{\partial u_{1y}(x, y, z, t)}{\partial y} - \frac{\partial u_{1z}(x, y, z, t)}{\partial z} \right] \right\} - \end{aligned}$$

$$-K\beta \frac{\partial T(x,y,z,t)}{\partial z}$$

Integration of left and right sides of the above equations on time t leads to the following results

$$\begin{aligned} u_{2x}(x,y,z,t) = & \int_0^t \int_0^g \frac{1}{\rho} \left[K + \frac{5E}{6(1+\gamma)} \right] \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2} d\tau d\vartheta + \int_0^t \int_0^g \frac{1}{\rho} \left[K - \frac{E}{3(1+\gamma)} \right] \times \\ & \times \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x \partial y} d\tau d\vartheta + \frac{1}{2} \int_0^t \int_0^g \frac{E}{1+\gamma} \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y^2} d\tau d\vartheta + \frac{1}{2} \int_0^t \int_0^g \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial z^2} \times \\ & \times \frac{E d\tau}{1+\gamma} d\vartheta + \int_0^t \int_0^g \frac{1}{\rho} \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial x \partial z} \left[K + \frac{E}{3(1+\sigma)} \right] d\tau d\vartheta - \int_0^t \int_0^g \frac{K\beta}{\rho} \frac{\partial T(x,y,z,\tau)}{\partial x} d\tau d\vartheta - \\ & - \int_0^t \int_0^g \frac{1}{\rho} \left[K + \frac{5E}{6(1+\sigma)} \right] \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2} d\tau d\vartheta - \int_0^t \int_0^g \frac{1}{\rho} \left[K - \frac{E}{3(1+\sigma)} \right] \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x \partial y} d\tau d\vartheta - \\ & - \frac{1}{2} \int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y^2} d\tau d\vartheta - \int_0^t \int_0^g \frac{1}{\rho(z)} \left[K + \frac{E}{3(1+\gamma)} \right] \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial x \partial z} d\tau d\vartheta - \\ & - \frac{1}{2} \int_0^t \int_0^g \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial z^2} \frac{E d\tau d\vartheta}{\rho(1+\gamma)} + \int_0^t \int_0^g \frac{K\beta}{\rho} \frac{\partial T(x,y,z,\tau)}{\partial x} d\tau d\vartheta + u_{0x} \\ u_{2y}(x,y,z,t) = & \frac{1}{2} \left[\int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2} d\tau d\vartheta + \int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x \partial y} d\tau d\vartheta \right] + \\ & + \int_0^t \int_0^g \frac{1}{\rho} \left[\frac{5E}{12(1+\gamma)} + K \right] \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial y^2} d\tau d\vartheta + \int_0^t \int_0^g \frac{K}{\rho} \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x \partial y} d\tau d\vartheta + \\ & + \frac{1}{2} \frac{\partial}{\partial z} \left[\int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial u_{1y}(x,y,z,\tau)}{\partial z} d\tau d\vartheta + \int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial u_{1z}(x,y,z,\tau)}{\partial y} d\tau d\vartheta \right] - \\ & - \int_0^t \int_0^g \frac{K\beta}{\rho} T(x,y,z,\tau) d\tau d\vartheta + \int_0^t \int_0^g \frac{1}{\rho} \left[K - \frac{E}{6(1+\gamma)} \right] \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y \partial z} d\tau d\vartheta - \\ & - \frac{1}{2} \left[\int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2} d\tau d\vartheta + \int_0^t \int_0^g \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x \partial y} d\tau d\vartheta \right] - \\ & - \int_0^t \int_0^g \frac{K\beta}{\rho} T(x,y,z,\tau) d\tau d\vartheta - \int_0^t \int_0^g \frac{K}{\rho} \frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x \partial y} d\tau d\vartheta - \int_0^t \int_0^g \left[K + \frac{5E}{12(1+\gamma)} \right] \times \end{aligned}$$

$$\begin{aligned}
 & \times \frac{1}{\rho} \frac{\partial^2 u_{1x}(x, y, z, \tau)}{\partial y^2} d\tau d\vartheta - \frac{1}{2} \frac{\partial}{\partial z} \left[\int_0^\vartheta \int_0^\vartheta \frac{E}{\rho(1+\gamma)} \frac{\partial u_{1y}(x, y, z, \tau)}{\partial z} d\tau d\vartheta + \int_0^\vartheta \int_0^\vartheta \frac{E}{\rho(1+\gamma)} \times \right. \\
 & \times \frac{\partial u_{1z}(x, y, z, \tau)}{\partial y} d\tau d\vartheta \left. \right] - \int_0^\vartheta \int_0^\vartheta \frac{1}{\rho} \left[K - \frac{E}{6(1+\gamma)} \right] \frac{\partial^2 u_{1y}(x, y, z, \tau)}{\partial y \partial z} d\tau d\vartheta + u_{0y} \\
 & u_z(x, y, z, t) = \frac{1}{2} \left[\int_0^\vartheta \int_0^\vartheta \frac{\partial^2 u_{1z}(x, y, z, \tau)}{\partial x^2} \frac{E d\tau d\vartheta}{\rho(1+\gamma)} + \int_0^\vartheta \int_0^\vartheta \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1z}(x, y, z, \tau)}{\partial y^2} d\tau d\vartheta + \right. \\
 & + \int_0^\vartheta \int_0^\vartheta \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1x}(x, y, z, \tau)}{\partial x \partial z} d\tau d\vartheta + \int_0^\vartheta \int_0^\vartheta \frac{E}{\rho(1+\gamma)} \frac{\partial^2 u_{1y}(x, y, z, \tau)}{\partial y \partial z} d\tau d\vartheta \left. \right] + \\
 & + \frac{\partial}{\partial z} \left[\int_0^\vartheta \int_0^\vartheta \frac{K}{\rho} \frac{\partial u_{1x}(x, y, z, \tau)}{\partial x} d\tau d\vartheta + \int_0^\vartheta \int_0^\vartheta \frac{K}{\rho} \frac{\partial u_{1x}(x, y, z, \tau)}{\partial y} d\tau d\vartheta + \int_0^\vartheta \int_0^\vartheta \frac{K}{\rho} \times \right. \\
 & \times \frac{\partial u_{1x}(x, y, z, \tau)}{\partial z} d\tau d\vartheta \left. \right] + \frac{\partial}{\partial z} \left[\int_0^\vartheta \int_0^\vartheta \frac{\partial u_{1z}(x, y, z, \tau)}{\partial z} \frac{E d\tau d\vartheta}{\rho(1+\gamma)} - \frac{1}{6} \int_0^\vartheta \int_0^\vartheta \frac{\partial u_{1x}(x, y, z, \tau)}{\partial x} \times \right. \\
 & \times \frac{E d\tau d\vartheta}{\rho(1+\gamma)} - \frac{1}{6} \int_0^\vartheta \int_0^\vartheta \frac{\partial u_{1y}(x, y, z, \tau)}{\partial y} \frac{E d\tau d\vartheta}{\rho(1+\gamma)} - \frac{1}{6} \int_0^\vartheta \int_0^\vartheta \frac{\partial u_{1z}(x, y, z, \tau)}{\partial z} \frac{E d\tau d\vartheta}{\rho(1+\gamma)} \left. \right] - \\
 & - \int_0^\vartheta \int_0^\vartheta \frac{K \beta}{\rho} \frac{\partial T(x, y, z, \tau)}{\partial z} d\tau d\vartheta + u_{0z}
 \end{aligned}$$

In the framework of this paper, we calculate components of displacement vector and spatio-temporal distribution of temperature by using the second-order approximation in the framework of method of averaging of function corrections. The approximation is usually enough good approximation to make qualitative analysis and to obtain some quantitative results. All obtained results have been checked by comparison with results of numerical simulations.

2. Discussion

In this section we analyzed changing of stress in metals during thermal treatment. Based on the analysis we estimate residual stress after finishing of the thermal treatment. Figure 1 shows typical dependences of components of displacement vector on coordinate. On this figure as an example, we consider dependence of component u_z on coordinate z . Increasing of number of curve and decreasing of coordinate correspond to increasing of defects of metals (dislocations, pores). The defects became as reasons residual stress in metals.

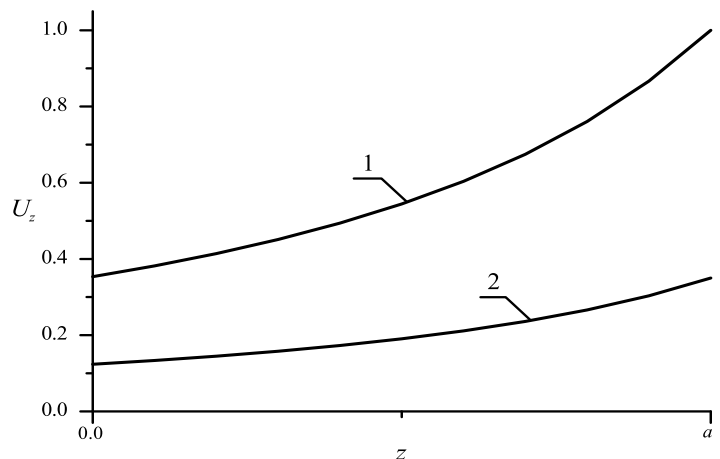


Fig. 1: Normalized dependences of component u_z of displacement vector on coordinate z . Increasing of number of curve and decreasing of coordinate correspond to increasing of defectiveness of metals.

3. Conclusion

In this paper we introduce an approach for estimation of stresses in metals during heat treatment and residual stresses in these materials after the treatment. In the framework of the considered approach, we introduce a model of changing of stress in metals during heat treatment as well as an analytical approach for its analysis.

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