

Periodic solutions for a Class of the generalized Hiemenz equation

Zouhair Diab^a, Juan L.G. Guirao^b

^aDepartment of Mathematics, Echahid Cheikh Larbi Tebessi University, 12002 Tebessa, Algeria

^bDepartment of Applied Mathematics and Statistics, Technical University of Cartagena, 30202 Cartagena, Spain

Abstract. In this paper, we investigate the zero-Hopf bifurcation in a class of generalized Hiemenz equations. By applying averaging theory, we derive explicit sufficient conditions for the existence of periodic solutions bifurcating from the zero-Hopf equilibrium at the origin. Furthermore, we establish conditions that guarantee the stability of the resulting periodic orbits.

1. Introduction and statement of the main results

Stagnation-point flows form a fundamental class of problems in fluid mechanics and applied mathematics. They describe the motion of a fluid in the neighborhood of a point or region where the velocity vanishes. Such flows are of particular interest because the stagnation region is associated with pronounced pressure and transport effects and occurs in a variety of configurations involving fluid impingement, solid surfaces, heat transfer, and mass transport [15, 20]. Their mathematical structure has also made them an important source of nonlinear differential equations arising from similarity reductions of the governing fluid equations.

The classical analysis of this type of flow originates from the work of Karl Hiemenz in 1911 [9]. By introducing an appropriate similarity representation, Hiemenz transformed the equations governing a steady two-dimensional viscous flow near a stagnation point into a nonlinear ordinary differential equation of third order. This reduction was important not only from the fluid-mechanical point of view, but also mathematically, since it provided a relatively simple nonlinear model retaining the essential characteristics of the original flow problem [12, 15].

The resulting classical Hiemenz equation can be written in the form

$$f''' + ff'' - f'^2 + 1 = 0, \quad (1)$$

supplemented with boundary conditions describing the behavior of the velocity field at the solid surface and away from it. The equation has subsequently become one of the classical models for planar stagnation-point flow. In particular, the Hiemenz solution provides an exact similarity solution of the governing

Corresponding author: Zouhair Diab, mail address: zouhair.diab@univ-tebessa.dz

Received: 18 August 2026; Accepted: 5 October 2026; Published: 20 October 2026

Keywords. Periodic solutions, Hiemenz equation, Ordinary differential equation

2010 *Mathematics Subject Classification.* 34C23, 34C25, 37G10

Cited this article as: Diab, Z., Guirao, J. L. G., Periodic solutions for a Class of the generalized Hiemenz equation, Engineering and Its Applications, 3(1), 2026, 1–10

© 2026 Engineering Journals. Published by Engineering Journals.

Navier–Stokes equations and can be studied independently as a nonlinear third-order boundary-value problem [12, 18].

The theory was subsequently extended beyond the original planar configuration. In 1936, Homann investigated the axisymmetric stagnation-point flow and obtained the corresponding formulation for a three-dimensional configuration [10]. Later, Howarth studied more general three-dimensional stagnation-point flows and showed that the governing problem can be reduced to a pair of coupled third-order ordinary differential equations involving a parameter related to the ratio of the principal strain rates [11]. In this sense, the developments of Homann and Howarth extended the original Hiemenz formulation from planar flow to more general three-dimensional geometries.

Further mathematical developments were obtained by Davey, who examined the saddle-point solutions associated with the three-dimensional equations introduced by Howarth [4]. In particular, the structure of the solutions depends strongly on the ratio between the relevant strain rates. Subsequent investigations by Davey and other authors contributed to a more detailed description of non-axisymmetric stagnation-point flows and their associated solution branches [4, 5]. These classical studies established a connection between the geometry of the underlying fluid flow and the qualitative structure of the solutions of the corresponding nonlinear ordinary differential equations.

The Hiemenz model has also provided a basis for numerous subsequent extensions. The classical stagnation-point configuration has been combined with stretching or shrinking surfaces, wall transpiration, permeability, heat and mass transfer, magnetic fields, rotation, and non-Newtonian constitutive effects [15, 19, 21]. Such extensions demonstrate that the original Hiemenz problem is not restricted to a single fluid-mechanical configuration; rather, it constitutes a useful mathematical framework from which a broad range of nonlinear flow models can be constructed. In particular, stagnation-point flows remain relevant in problems involving high-pressure regions and enhanced heat-transfer rates near the stagnation location [20, 21].

From the perspective of nonlinear dynamics, the importance of the Hiemenz-type equations goes beyond their original interpretation as boundary-layer models. Since the governing equation is of third order, it can be converted naturally into a three-dimensional autonomous system by introducing the first and second derivatives of the dependent variable as additional state variables. This formulation makes it possible to investigate the associated phase space, equilibria, stability properties, invariant structures, and periodic trajectories by means of methods from the qualitative theory of differential equations.

This dynamical interpretation has motivated the construction of generalized Hiemenz-type equations. For example, Uribe and Martínez considered a ten-parameter third-order differential equation derived from a generalization of the equation associated with Hiemenz flow after omitting the boundary conditions [17]. Their formulation illustrates how the classical Hiemenz model can be embedded into a broader family of nonlinear autonomous differential equations. Such generalizations provide a convenient framework for investigating dynamical phenomena that are not directly accessible from the original boundary-value formulation.

A zero-Hopf bifurcation is one of the mechanisms responsible for the appearance of periodic orbits from a nonhyperbolic equilibrium. For a three-dimensional autonomous system, an equilibrium P is said to be of zero-Hopf type when the eigenvalues of the Jacobian matrix at P are $0, i\omega$, and $i\omega$, with $\omega \neq 0$. This configuration has been considered for several classes of three-dimensional polynomial systems by means of averaging theory [7, 14]. In particular, Diab, Guirao, and Vera studied the zero-Hopf bifurcation in a generalized Genesio equation and established the existence of periodic trajectories emerging from the corresponding equilibrium [7]. The study of this phenomenon is not restricted to dimension three. Diab, Guirao, and Llibre considered a five-dimensional quadratic polynomial system and, using averaging theory, obtained two families of periodic orbits associated with a zero-Hopf equilibrium [6].

Motivated by these results, we investigate the zero-Hopf configuration for the generalized Hiemenz equation

$$\ddot{x} + (a_1\dot{x} + b_1x^2 + c_1)\ddot{x} + (a_2\ddot{x} + b_2\dot{x}^2 + c_2x + d_2)\dot{x} + (a_3\dot{x}^2 + b_3\dot{x} + c_3)x = 0, \quad (2)$$

where all parameters $a_1, b_1, c_1, a_2, b_2, c_2, d_2, a_3, b_3$ and c_3 are real.

Letting $y = \dot{x}$, $z = \dot{y}$ and $\dot{z} = w$, the differential equation (2) can be transformed into the following system of first order differential equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -(a_1 y + b_1 x^2 + c_1)z - (a_2 z + b_2 y^2 + c_2 x + d_2)y - (a_3 z^2 + b_3 y + c_3)x, \end{cases} \quad (3)$$

The equilibria of system (3) are $O = (0, 0, 0)$ and $E = (x, 0, 0)$, whenever the corresponding equilibrium conditions are satisfied. We determine the conditions on the parameters for which the origin is a zero-Hopf equilibrium. A suitable perturbation of the parameters is then introduced, and averaging theory is applied to the resulting system. This allows us to derive explicit conditions for the existence of the periodic solutions bifurcating from the origin.

Proposition 1.1. *Suppose that $c_1 = 0$, $c_3 = 0$ and $d_2 > 0$ hold. Then the equilibrium $O = (0, 0, 0)$ is a zero-Hopf equilibrium of the generalized Hiemenz system (3).*

Theorem 1.2. *Suppose that*

$$c_1 = \varepsilon^2 a, \quad c_3 = \varepsilon^2 c \quad \text{and} \quad d_2 = \omega^2 + \varepsilon b,$$

with $\omega > 0$ and ε a sufficiently small parameter. Then, the generalized Hiemenz system (3) has a zero-Hopf bifurcation at the equilibrium point $O = (0, 0, 0)$, and at most three periodic orbits can bifurcate from this equilibrium when $\varepsilon = 0$ if

$$\frac{(4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1)(a\omega^2 - c)}{3\omega^6(\omega^2 a_3 - b_1)} \neq 0,$$

and

$$\frac{(4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1)c}{\omega^6(\omega^2 a_3 - 2b_1)} \neq 0,$$

Furthermore, the stability of these periodic orbits is determined by the real parts of the corresponding characteristic quantities. If

$$\operatorname{Re}\left(-\frac{a\omega^2 - c}{\omega^3}\right) < 0,$$

and

$$\operatorname{Re}\left(\frac{4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1}{3\omega^3(-\omega^2 a_3 + b_1)}\right) < 0,$$

then one of the three periodic orbits is stable. The other two are stable whenever

$$\operatorname{Re}\left(\frac{3\omega^2 c a_3 - 3c b_1 + \sqrt{25\omega^4 c^2 a_3^2 - 34\omega^2 c^2 a_3 b_1 - 23c^2 b_1^2 - 128\omega^4 a_3 c b_1 a + 32\omega^6 a_3^2 c a + 128c\omega^2 b_1^2 a}}{4(\omega^2 a_3 - 2b_1)\omega^3}\right) < 0,$$

and

$$\operatorname{Re}\left(\frac{3\omega^2 c a_3 - 3c b_1 - \sqrt{25\omega^4 c^2 a_3^2 - 34\omega^2 c^2 a_3 b_1 - 23c^2 b_1^2 - 128\omega^4 a_3 c b_1 a + 32\omega^6 a_3^2 c a + 128c\omega^2 b_1^2 a}}{4(\omega^2 a_3 - 2b_1)\omega^3}\right) < 0.$$

2. Preliminaries

2.1. The averaging theory of first and second order

Averaging theory of the first and second orders is employed to investigate the existence of periodic solutions; the reader is referred to [3, 13] for a detailed presentation of the corresponding theoretical framework. This approach constitutes a well-established analytical method for studying the qualitative behavior of smooth nonlinear differential systems, particularly in the detection and analysis of periodic trajectories. The first formalization of this procedure is due to Fatou [8] in 1928. Important practical and theoretical contributions in this theory were made by Krylov and Bogoliubov [2] in the 1930s and Bogoliubov [1] in 1945.

Theorem 2.1. Consider the differential system

$$\dot{x} = \varepsilon \mathbf{F}(t, x) + \varepsilon^2 \mathbf{G}(t, x) + \varepsilon^3 \mathbf{R}(t, x, \varepsilon), \quad x(0) = x_0, \quad (4)$$

where $\mathbf{F}, \mathbf{G} : \mathbb{R} \times D \rightarrow \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ are continuous functions, T -periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

- (i) $\mathbf{F}(t, \cdot), \mathbf{G}(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, $\mathbf{F}, \mathbf{G}, \mathbf{R}, D_x \mathbf{F}$ and $D_x \mathbf{G}$ are locally Lipschitz with respect to x , and $\mathbf{R}$ is differentiable with respect to ε . We define $\mathbf{f}, \mathbf{g} : D \rightarrow \mathbb{R}^n$ as

$$\begin{aligned} \mathbf{f}(z) &= \frac{1}{T} \int_0^T \mathbf{F}(s, z) ds, \\ \mathbf{g}(z) &= \frac{1}{T} \int_0^T \left[D_z \mathbf{F}(s, z) \int_0^s \mathbf{F}(t, z) dt + \mathbf{G}(s, z) \right] ds. \end{aligned}$$

- (ii) For $V \subset D$, an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $p \in V$ such that $\mathbf{f}(p) + \varepsilon \mathbf{g}(p) = 0$ and

$$\det \left(\frac{\partial(\mathbf{f} + \varepsilon \mathbf{g})}{\partial z} \Big|_{z=p} \right) \neq 0,$$

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T -periodic solution $\varphi(t, \varepsilon)$ of system (4) such that $\varphi(0, \varepsilon) \rightarrow p$ when $\varepsilon \rightarrow 0$. If the function $\mathbf{f}$ is not identically zero, then the zeros of $\mathbf{f} + \varepsilon \mathbf{g}$ are mainly the zeros of $\mathbf{f}$ for ε sufficiently small. In this case, Theorem 2.1 provides the so-called averaging theory of first order. If the function $\mathbf{f}$ is identically zero and $\mathbf{g}$ is not identically zero, then the zeros of $\mathbf{f} + \varepsilon \mathbf{g}$ are the zeros of $\mathbf{g}$. In this case, Theorem 2.1 provides the so-called averaging theory of second order. In the case of the averaging theory of first order, we consider in D the averaged differential equation

$$\dot{y} = \varepsilon \mathbf{f}(y), \quad y(0) = x_0, \quad (5)$$

where

$$\mathbf{f}(y) = \frac{1}{T} \int_0^T \mathbf{F}(t, y) dt$$

Then Theorem 2.1 gives us information about the stability or instability of the limit cycle $\varphi(t, \varepsilon)$. In fact, it is given by the stability or instability of the equilibrium point p of the averaged system (5). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$. In the case of the averaging theory of second order, that is, $\mathbf{f} \equiv 0$ and $\mathbf{g}$ non-identically zero, we have that the stability and instability of the limit cycle $\varphi(t, \varepsilon)$ coincide with the type of stability or instability of the equilibrium point p of the averaged system

$$\dot{y} = \varepsilon^2 \mathbf{g}(y), \quad y(0) = x_0,$$

that is, it is the same that the singular point p associated the Poincaré map of the limit cycle $\varphi(t, \varepsilon)$. For additional information on averaging theory, see the book [16].

3. Proof of Proposition 1.1 and Theorem 1.2

Proof. [Proof of Proposition 1.1] The characteristic polynomial $p(\lambda)$ of the linear part of the generalized Hiemenz system at the equilibrium point located at the origin O is given by

$$p(\lambda) = \lambda^3 + c_1 \lambda^2 + d_2 \lambda + c_3,$$

Imposing that $p(\lambda) = \lambda(\lambda^2 + \omega^2)$, with $\omega > 0$. Then, we obtain $c_1 = 0, c_3 = 0$ and $d_2 = \omega^2 > 0$. This completes the Proof of Proposition 1.1. $\square$

Proof. [Proof of Theorem 1.2.] Let us consider $c_1 = \varepsilon^2 a$, $c_3 = \varepsilon^2 c$ and $d_2 = \omega^2 + \varepsilon b$, being $\omega > 0$ and $\varepsilon > 0$ a sufficiently small parameter. Then, the system (3) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -(a_1 y + b_1 x^2)z - (a_2 z + b_2 y^2 + c_2 x + \omega^2)y \\ \quad - (a_3 z^2 + b_3 y)x - by\varepsilon - (cx + az)\varepsilon^2, \end{cases} \quad (6)$$

The first step in order to write our differential system (6) in the normal form for applying the averaging theory is to write the linear part at the origin of system (6) when $\varepsilon = 0$ into its real Jordan normal form, that is into the form

$$\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

After the linear change of variables

$$(x, y, z) \rightarrow \left(-\frac{X}{\omega^2} + Z, -\frac{Y}{\omega}, X\right), \quad (7)$$

then system (6) in the new variables (X, Y, Z) can be rewritten as

$$\begin{aligned} \dot{X} &= (a_3 X^3 \omega^2 - a_3 X^2 \omega^4 Z - \omega b_3 YX + b_3 YZ \omega^3 + X a_1 Y \omega^3 - b_1 X^3 + 2 b_1 X^2 Z \omega^2 \\ &\quad - X b_1 Z^2 \omega^4 + Y \omega^3 a_2 X + Y^3 \omega b_2 - Y \omega c_2 X + Y \omega^3 c_2 Z + Y \omega^5)/\omega^4 \\ &\quad + bY\varepsilon/\omega + (-\omega^4 aX + \omega^2 cX - \omega^4 cZ)\varepsilon^2/\omega^4, \\ \dot{Y} &= -\omega X, \\ \dot{Z} &= (-a_3 X^2 \omega^4 Z - \omega b_3 YX + b_3 YZ \omega^3 - X b_1 Z^2 \omega^4 + X a_1 Y \omega^3 - b_1 X^3 + 2 b_1 X^2 Z \omega^2 \\ &\quad + Y \omega^3 c_2 Z + Y \omega^3 a_2 X + Y^3 \omega b_2 - Y \omega c_2 X + a_3 X^3 \omega^2)/\omega^6 + bY\varepsilon/\omega^3 \\ &\quad + (-\omega^4 aX + \omega^2 cX - \omega^4 cZ)\varepsilon^2/\omega^6. \end{aligned} \quad (8)$$

By means of the rescaling of variables given by $(X, Y, Z) = (\varepsilon X_1, \varepsilon Y_1, \varepsilon Z_1)$, the system (8) becomes

$$\begin{aligned} \dot{X}_1 &= \omega Y_1 - \left(-X_1 a_1 Y_1 \omega^3 + Y_1 \omega c_2 X_1 - Y_1 \omega^3 c_2 Z_1 - Y_1 \omega^3 a_2 X_1 \right. \\ &\quad \left. + \omega b_3 Y_1 X_1 - b_3 Y_1 Z_1 \omega^3 - bY_1 \omega^3\right)\varepsilon/\omega^4 \\ &\quad - (a_3 X_1^2 \omega^4 Z_1 + b_1 X_1^3 - 2 b_1 X_1^2 Z_1 \omega^2 + X_1 b_1 Z_1^2 \omega^4 + \omega^4 aX_1 \\ &\quad - Y_1^3 \omega b_2 - \omega^2 cX_1 + \omega^4 cZ_1 - a_3 X_1^3 \omega^2)\varepsilon^2/\omega^4, \\ \dot{Y}_1 &= -\omega X_1, \\ \dot{Z}_1 &= -\left(-X_1 a_1 Y_1 \omega^3 + Y_1 \omega c_2 X_1 - Y_1 \omega^3 c_2 Z_1 - Y_1 \omega^3 a_2 X_1 + \omega b_3 Y_1 X_1 \right. \\ &\quad \left. - b_3 Y_1 Z_1 \omega^3 - bY_1 \omega^3\right)\varepsilon/\omega^6, \\ &\quad - (a_3 X_1^2 \omega^4 Z_1 + b_1 X_1^3 - 2 b_1 X_1^2 Z_1 \omega^2 + X_1 b_1 Z_1^2 \omega^4 + \omega^4 aX_1 \\ &\quad - Y_1^3 \omega b_2 - \omega^2 cX_1 + \omega^4 cZ_1 - a_3 X_1^3 \omega^2)\varepsilon^2/\omega^6, \end{aligned} \quad (9)$$

Performing the change of variables to cylindrical coordinates $(X_1, Y_1, Z_1) \mapsto (r, \theta, Z_1)$ with $X_1 = r \cos \theta$,

$Y_1 = r \sin \theta$ and $Z_1 = Z_1$. Then the system (9) can be written as follows

$$\begin{aligned}
 \dot{r} = & -\cos(\theta) \left(-r^2 \cos(\theta) a_1 \sin(\theta) \omega^3 + r^2 \sin(\theta) \omega c_2 \cos(\theta) - r \sin(\theta) \omega^3 c_2 Z_1 \right. \\
 & -r^2 \sin(\theta) \omega^3 a_2 \cos(\theta) + \omega b_3 r^2 \sin(\theta) \cos(\theta) - b_3 r \sin(\theta) Z_1 \omega^3 \\
 & \left. -br \sin(\theta) \omega^3 \right) \varepsilon / \omega^4 - \cos(\theta) \left(-a_3 r^3 (\cos(\theta))^3 \omega^2 \right. \\
 & +a_3 r^2 (\cos(\theta))^2 \omega^4 Z_1 + r \cos(\theta) b_1 Z_1^2 \omega^4 + \omega^4 ar \cos(\theta) \\
 & -2 b_1 r^2 (\cos(\theta))^2 Z_1 \omega^2 - \omega^2 cr \cos(\theta) + b_1 r^3 (\cos(\theta))^3 \\
 & \left. +\omega^4 c Z_1 - \sin(\theta) r^3 \omega b_2 + \sin(\theta) r^3 \omega b_2 (\cos(\theta))^2 \right) \varepsilon^2 / \omega^4 \\
 \dot{\theta} = & -\omega - \left(r^2 \omega^3 a_2 \cos(\theta) - (\cos(\theta))^3 r^2 \omega^3 a_2 - (\cos(\theta))^3 r^2 \omega^3 a_1 \right. \\
 & -(\cos(\theta))^2 r \omega^3 c_2 Z_1 - (\cos(\theta))^2 r \omega^3 b + r b \omega^3 + (\cos(\theta))^3 r^2 \omega b_3 \\
 & -r^2 \omega c_2 \cos(\theta) + (\cos(\theta))^3 r^2 \omega c_2 - (\cos(\theta))^2 r \omega^3 b_3 Z_1 \\
 & \left. +r^2 \cos(\theta) a_1 \omega^3 + r \omega^3 c_2 Z_1 + r b_3 Z_1 \omega^3 - \omega b_3 r^2 \cos(\theta) \right) \varepsilon / r \omega^4 \\
 & -\left(\sin(\theta) \omega^2 cr \cos(\theta) + \sin(\theta) a_3 r^3 (\cos(\theta))^3 \omega^2 - \sin(\theta) b_1 r^3 (\cos(\theta))^3 \right. \\
 & +r^3 \omega b_2 - \sin(\theta) \omega^4 ar \cos(\theta) - \sin(\theta) a_3 r^2 (\cos(\theta))^2 \omega^4 Z_1 + (\cos(\theta))^4 r^3 \omega b_2 \\
 & +2 \sin(\theta) b_1 r^2 (\cos(\theta))^2 Z_1 \omega^2 - \sin(\theta) r \cos(\theta) b_1 Z_1^2 \omega^4 - 2 r^3 \omega b_2 (\cos(\theta))^2 \\
 & \left. - \sin(\theta) \omega^4 c Z_1 \right) \varepsilon^2 / r \omega^4, \\
 \dot{Z}_1 = & -\left(-r^2 \cos(\theta) a_1 \sin(\theta) \omega^3 + r^2 \sin(\theta) \omega c_2 \cos(\theta) - r \sin(\theta) \omega^3 c_2 Z_1 \right. \\
 & -r^2 \sin(\theta) \omega^3 a_2 \cos(\theta) + \omega b_3 r^2 \sin(\theta) \cos(\theta) - b_3 r \sin(\theta) Z_1 \omega^3 \\
 & -br \sin(\theta) \omega^3 \left. \right) \varepsilon / \omega^6 - \left(-a_3 r^3 (\cos(\theta))^3 \omega^2 + a_3 r^2 (\cos(\theta))^2 \omega^4 Z_1 \right. \\
 & +r \cos(\theta) b_1 Z_1^2 \omega^4 + \omega^4 ar \cos(\theta) - 2 b_1 r^2 (\cos(\theta))^2 Z_1 \omega^2 \\
 & -\omega^2 cr \cos(\theta) + b_1 r^3 (\cos(\theta))^3 + \omega^4 c Z_1 - \sin(\theta) r^3 \omega b_2 \\
 & \left. +\sin(\theta) r^3 \omega b_2 (\cos(\theta))^2 \right) \varepsilon^2 / \omega^6,
 \end{aligned} \tag{10}$$

In (10) we consider new independent variable, and we obtain

$$\begin{aligned}
 \frac{dr}{d\theta} &= \varepsilon \mathbf{F}_1(\theta, r, Z_1) + \varepsilon^2 \mathbf{G}_1(\theta, r, Z_1) + \mathcal{O}(\varepsilon^3), \\
 \frac{dZ_1}{d\theta} &= \varepsilon \mathbf{F}_2(\theta, r, Z_1) + \varepsilon^2 \mathbf{G}_2(\theta, r, Z_1) + \mathcal{O}(\varepsilon^3),
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 \mathbf{F}_1(\theta, r, Z_1) &= -\cos(\theta) r \sin(\theta) \left(r \cos(\theta) a_1 \omega^2 - r c_2 \cos(\theta) + \omega^2 c_2 Z_1 + r \omega^2 a_2 \cos(\theta) \right. \\
 &\quad \left. -b_3 r \cos(\theta) + b_3 Z_1 \omega^2 + b \omega^2 \right) / \omega^4, \\
 \mathbf{F}_2(\theta, r, Z_1) &= -r \sin(\theta) \left(r \cos(\theta) a_1 \omega^2 - r c_2 \cos(\theta) + \omega^2 c_2 Z_1 + r \omega^2 a_2 \cos(\theta) \right. \\
 &\quad \left. -b_3 r \cos(\theta) + b_3 Z_1 \omega^2 + b \omega^2 \right) / \omega^6,
 \end{aligned}$$

and

$$\begin{aligned}
\mathbf{G}_1(\theta, r, Z_1) = & -\cos(\theta) \left(-\sin(\theta) r b^2 \omega^4 + \omega^5 a_3 r^3 (\cos(\theta))^3 + \sin(\theta) r^3 b_3^2 (\cos(\theta))^4 \right. \\
& -\sin(\theta) r^3 b_3^2 (\cos(\theta))^2 + \sin(\theta) r^3 c_2^2 (\cos(\theta))^4 - \omega^3 b_1 r^3 (\cos(\theta))^3 \\
& + \omega^4 \sin(\theta) r^3 b_2 + \omega^5 c r \cos(\theta) - \sin(\theta) r^3 c_2^2 (\cos(\theta))^2 \\
& - \omega^7 a r \cos(\theta) - 4 \sin(\theta) r^2 c_2 (\cos(\theta))^3 \omega^2 b_3 Z_1 \\
& - 2 \sin(\theta) r^2 \cos(\theta) a_1 \omega^4 b_3 Z_1 - 2 \sin(\theta) r^2 \cos(\theta) a_1 \omega^4 c_2 Z_1 \\
& + 2 \sin(\theta) r^2 (\cos(\theta))^3 a_1 \omega^4 c_2 Z_1 + 2 \sin(\theta) r^2 (\cos(\theta))^3 a_1 \omega^4 b_3 Z_1 \\
& + 4 \sin(\theta) r^2 c_2 \cos(\theta) b_3 Z_1 \omega^2 - \omega^7 c Z_1 + \sin(\theta) r \omega^4 c_2^2 Z_1^2 (\cos(\theta))^2 \\
& + \sin(\theta) r b^2 \omega^4 (\cos(\theta))^2 - \omega^4 \sin(\theta) r^3 b_2 (\cos(\theta))^2 - \sin(\theta) r^3 (\cos(\theta))^2 a_1^2 \omega^4 \\
& - \omega^7 a_3 r^2 (\cos(\theta))^2 Z_1 - \omega^7 r \cos(\theta) b_1 Z_1^2 + 2 \sin(\theta) r^3 (\cos(\theta))^2 a_1 \omega^2 b_3 \\
& - 2 \sin(\theta) r^3 (\cos(\theta))^4 a_1 \omega^2 c_2 - 2 \sin(\theta) r^3 (\cos(\theta))^4 a_1 \omega^2 b_3 \\
& - 2 \sin(\theta) r^2 \cos(\theta) a_1 \omega^4 b - 2 \sin(\theta) r \omega^4 c_2 Z_1^2 b_3 - 2 \sin(\theta) r b_3 Z_1 \omega^4 b \\
& - 2 \sin(\theta) r^3 c_2 (\cos(\theta))^2 b_3 - 2 \sin(\theta) r \omega^4 c_2 Z_1 b \\
& - 2 \sin(\theta) r^2 b_3 (\cos(\theta))^3 \omega^2 b + 2 \sin(\theta) r^3 \omega^2 a_2 (\cos(\theta))^2 b_3 \\
& + 2 \sin(\theta) r^2 b_3 \cos(\theta) b \omega^2 - 2 \sin(\theta) r^3 c_2 (\cos(\theta))^4 \omega^2 a_2 \\
& - 2 \sin(\theta) r^2 c_2^2 (\cos(\theta))^3 \omega^2 Z_1 - 2 \sin(\theta) r^2 \omega^4 a_2 \cos(\theta) b \\
& - 2 \sin(\theta) r^3 \omega^2 a_2 (\cos(\theta))^4 b_3 + 2 \sin(\theta) r^3 (\cos(\theta))^2 a_1 \omega^2 c_2 \\
& + 2 \sin(\theta) r^2 c_2^2 \cos(\theta) \omega^2 Z_1 + 2 \sin(\theta) r^2 (\cos(\theta))^3 a_1 \omega^4 b \\
& + 2 \sin(\theta) r^3 (\cos(\theta))^4 a_1 \omega^4 a_2 + 2 \sin(\theta) r^2 \omega^4 a_2 (\cos(\theta))^3 b \\
& + 2 \sin(\theta) r^2 b_3^2 \cos(\theta) Z_1 \omega^2 + 2 \sin(\theta) r^3 c_2 (\cos(\theta))^2 \omega^2 a_2 \\
& + 2 \omega^5 b_1 r^2 (\cos(\theta))^2 Z_1 - 2 \sin(\theta) r^2 b_3^2 (\cos(\theta))^3 \omega^2 Z_1 \\
& - 2 \sin(\theta) r^2 c_2 (\cos(\theta))^3 \omega^2 b + \sin(\theta) r^3 (\cos(\theta))^4 a_1^2 \omega^4 \\
& + 2 \sin(\theta) r^2 c_2 \cos(\theta) b \omega^2 - 2 \sin(\theta) r^3 (\cos(\theta))^2 a_1 \omega^4 a_2 \\
& + 2 \sin(\theta) r^3 c_2 (\cos(\theta))^4 b_3 - \sin(\theta) r^3 \omega^4 a_2^2 (\cos(\theta))^2 \\
& + \sin(\theta) r^3 \omega^4 a_2^2 (\cos(\theta))^4 + \sin(\theta) r b_3^2 Z_1^2 \omega^4 (\cos(\theta))^2 \\
& - \sin(\theta) r \omega^4 c_2^2 Z_1^2 - \sin(\theta) r b_3^2 Z_1^2 \omega^4 - 2 \sin(\theta) r^2 \omega^4 c_2 Z_1 a_2 \cos(\theta) \\
& + 2 \sin(\theta) r^2 \omega^4 c_2 Z_1 (\cos(\theta))^3 a_2 + 2 \sin(\theta) r \omega^4 c_2 Z_1 (\cos(\theta))^2 b \\
& + 2 \sin(\theta) r \omega^4 c_2 Z_1^2 (\cos(\theta))^2 b_3 + 2 \sin(\theta) r^2 \omega^4 a_2 (\cos(\theta))^3 b_3 Z_1 \\
& \left. - 2 \sin(\theta) r^2 \omega^4 a_2 \cos(\theta) b_3 Z_1 + 2 \sin(\theta) r b_3 Z_1 \omega^4 (\cos(\theta))^2 b \right) / \omega^8
\end{aligned}$$

$$\begin{aligned}
\mathbf{G}_2(\theta, r, Z_1) = & -\left(-\sin(\theta)rb^2\omega^4 + \omega^5a_3r^3(\cos(\theta))^3 + \sin(\theta)r^3b_3^2(\cos(\theta))^4\right. \\
& -\sin(\theta)r^3b_3^2(\cos(\theta))^2 + \sin(\theta)r^3c_2^2(\cos(\theta))^4 - \omega^3b_1r^3(\cos(\theta))^3 \\
& +\omega^4\sin(\theta)r^3b_2 + \omega^5cr\cos(\theta) - \sin(\theta)r^3c_2^2(\cos(\theta))^2 \\
& -\omega^7ar\cos(\theta) - 4\sin(\theta)r^2c_2(\cos(\theta))^3\omega^2b_3Z_1 \\
& -2\sin(\theta)r^2\cos(\theta)a_1\omega^4b_3Z_1 - 2\sin(\theta)r^2\cos(\theta)a_1\omega^4c_2Z_1 \\
& +2\sin(\theta)r^2(\cos(\theta))^3a_1\omega^4c_2Z_1 + 2\sin(\theta)r^2(\cos(\theta))^3a_1\omega^4b_3Z_1 \\
& +4\sin(\theta)r^2c_2\cos(\theta)b_3Z_1\omega^2 - \omega^7cZ_1 + \sin(\theta)r\omega^4c_2^2Z_1^2(\cos(\theta))^2 \\
& +\sin(\theta)rb^2\omega^4(\cos(\theta))^2 - \omega^4\sin(\theta)r^3b_2(\cos(\theta))^2 \\
& -\sin(\theta)r^3(\cos(\theta))^2a_1^2\omega^4 - \omega^7a_3r^2(\cos(\theta))^2Z_1 - \omega^7r\cos(\theta)b_1Z_1^2 \\
& +2\sin(\theta)r^3(\cos(\theta))^2a_1\omega^2b_3 - 2\sin(\theta)r^3(\cos(\theta))^4a_1\omega^2c_2 \\
& -2\sin(\theta)r^3(\cos(\theta))^4a_1\omega^2b_3 - 2\sin(\theta)r^2\cos(\theta)a_1\omega^4b \\
& -2\sin(\theta)r\omega^4c_2Z_1^2b_3 - 2\sin(\theta)rb_3Z_1\omega^4b - 2\sin(\theta)r^3c_2(\cos(\theta))^2b_3 \\
& -2\sin(\theta)r\omega^4c_2Z_1b - 2\sin(\theta)r^2b_3(\cos(\theta))^3\omega^2b \\
& +2\sin(\theta)r^3\omega^2a_2(\cos(\theta))^2b_3 + 2\sin(\theta)r^2b_3\cos(\theta)b\omega^2 \\
& -2\sin(\theta)r^3c_2(\cos(\theta))^4\omega^2a_2 - 2\sin(\theta)r^2c_2^2(\cos(\theta))^3\omega^2Z_1 \\
& -2\sin(\theta)r^2\omega^4a_2\cos(\theta)b - 2\sin(\theta)r^3\omega^2a_2(\cos(\theta))^4b_3 \\
& +2\sin(\theta)r^3(\cos(\theta))^2a_1\omega^2c_2 + 2\sin(\theta)r^2c_2^2\cos(\theta)\omega^2Z_1 \\
& +2\sin(\theta)r^2(\cos(\theta))^3a_1\omega^4b + 2\sin(\theta)r^3(\cos(\theta))^4a_1\omega^4a_2 \\
& +2\sin(\theta)r^2\omega^4a_2(\cos(\theta))^3b + 2\sin(\theta)r^2b_3^2\cos(\theta)Z_1\omega^2 \\
& +2\sin(\theta)r^3c_2(\cos(\theta))^2\omega^2a_2 + 2\omega^5b_1r^2(\cos(\theta))^2Z_1 \\
& -2\sin(\theta)r^2b_3^2(\cos(\theta))^3\omega^2Z_1 - 2\sin(\theta)r^2c_2(\cos(\theta))^3\omega^2b \\
& +\sin(\theta)r^3(\cos(\theta))^4a_1^2\omega^4 + 2\sin(\theta)r^2c_2\cos(\theta)b\omega^2 \\
& -2\sin(\theta)r^3(\cos(\theta))^2a_1\omega^4a_2 + 2\sin(\theta)r^3c_2(\cos(\theta))^4b_3 \\
& -\sin(\theta)r^3\omega^4a_2^2(\cos(\theta))^2 + \sin(\theta)r^3\omega^4a_2^2(\cos(\theta))^4 \\
& +\sin(\theta)rb_3^2Z_1^2\omega^4(\cos(\theta))^2 - \sin(\theta)r\omega^4c_2^2Z_1^2 \\
& -\sin(\theta)rb_3^2Z_1^2\omega^4 - 2\sin(\theta)r^2\omega^4c_2Z_1a_2\cos(\theta) \\
& +2\sin(\theta)r^2\omega^4c_2Z_1(\cos(\theta))^3a_2 + 2\sin(\theta)r\omega^4c_2Z_1(\cos(\theta))^2b \\
& +2\sin(\theta)r\omega^4c_2Z_1^2(\cos(\theta))^2b_3 + 2\sin(\theta)r^2\omega^4a_2(\cos(\theta))^3b_3Z_1 \\
& \left.-2\sin(\theta)r^2\omega^4a_2\cos(\theta)b_3Z_1 + 2\sin(\theta)rb_3Z_1\omega^4(\cos(\theta))^2b\right)/\omega^{10}
\end{aligned}$$

With the notation introduced at subsection 2.1, is $t = \theta, T = 2\pi, x = (r, Z_1)^T$ and

$$\begin{aligned}
\mathbf{F}(\theta, r, Z_1) &= \begin{pmatrix} \mathbf{F}_1(\theta, r, Z_1) \\ \mathbf{F}_2(\theta, r, Z_1) \end{pmatrix} \quad \text{and} \quad \mathbf{G}(\theta, r, Z_1) = \begin{pmatrix} \mathbf{G}_1(\theta, r, Z_1) \\ \mathbf{G}_2(\theta, r, Z_1) \end{pmatrix}. \\
\mathbf{f}(r, Z_1) &= \begin{pmatrix} \mathbf{f}_1(r, Z_1) \\ \mathbf{f}_2(r, Z_1) \end{pmatrix} \quad \text{and} \quad \mathbf{g}(r, Z_1) = \begin{pmatrix} \mathbf{g}_1(r, Z_1) \\ \mathbf{g}_2(r, Z_1) \end{pmatrix}.
\end{aligned}$$

Easily can be checked that (12) satisfies all the assumptions of Theorem 2.1.

We calculate the first-order averaging function $\mathbf{f}(r, Z_1)$ obtaining

$$\mathbf{f}(r, Z_1) = \begin{pmatrix} \mathbf{f}_1(r, Z_1) \\ \mathbf{f}_2(r, Z_1) \end{pmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{F}(s, r, Z_1) ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We pass then to the averaging of second order and we compute the second-order averaging function $\mathbf{g}(r, Z_1)$

$$\begin{aligned}\mathbf{g}(r, Z_1) &= \begin{pmatrix} \mathbf{g}_1(r, Z_1) \\ \mathbf{g}_2(r, Z_1) \end{pmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \left[D_{(r, Z_1)} \mathbf{F}(s, r, Z_1) \int_0^s \mathbf{F}(t, r, Z_1) dt + \mathbf{G}(s, r, Z_1) \right] ds, \\ &= \begin{pmatrix} r(-3\omega^2 a_3 r^2 - 4\omega^2 c + 4\omega^4 a + 3b_1 r^2 + 4\omega^4 b_1 Z_1^2) / 8\omega^5 \\ Z_1(\omega^2 a_3 r^2 + 2\omega^2 c - 2b_1 r^2) / 2\omega^5 \end{pmatrix}\end{aligned}\quad (12)$$

By solving the system of nonlinear equations $\mathbf{g}_1(r, Z_1) = \mathbf{g}_2(r, Z_1) = 0$, we obtain the following four solutions

$$\begin{aligned}\mathbf{S}_0 &= (0, 0), \quad \mathbf{S}_1 = (r_1^*, Z_1^*), \\ \mathbf{S}_2 &= (r_2^*, Z_2^*), \quad \mathbf{S}_3 = (r_2^*, -Z_2^*),\end{aligned}$$

where

$$\begin{aligned}r_1^* &= \sqrt{\frac{4a\omega^2 - 4c}{3\omega^2 a_3 - 3b_1}} \omega, \\ r_2^* &= \sqrt{-\frac{2c}{\omega^2 a_3 - 2b_1}} \omega, \\ Z_1^* &= 0,\end{aligned}$$

and

$$Z_2^* = \frac{1}{\omega} \sqrt{\frac{4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1}{2\omega^2 a_3 b_1 - 4b_1^2}},$$

The solution $\mathbf{S}_0$ corresponds to the equilibrium at the origin, so it is not a good solution.

For other solutions, we get

$$\det(D_{\mathbf{y}} \mathbf{g}(\mathbf{S}_1)) = \frac{(4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1)(a\omega^2 - c)}{3\omega^6 (\omega^2 a_3 - b_1)},$$

$$\det(D_{\mathbf{y}} \mathbf{g}(\mathbf{S}_{2,3})) = \frac{(4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1)c}{\omega^6 (\omega^2 a_3 - 2b_1)},$$

Since by assumptions

$$\frac{(4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1)(a\omega^2 - c)}{3\omega^6 (\omega^2 a_3 - b_1)} \neq 0,$$

and

$$\frac{(4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1)c}{\omega^6 (\omega^2 a_3 - 2b_1)} \neq 0,$$

the solutions $\mathbf{S}_i$ exist and $\det(D_{\mathbf{y}} \mathbf{g}(\mathbf{S}_i)) \neq 0$ for $i = 1, 2, 3$. From Theorem 2.1, it follows that system (11) for $\varepsilon > 0$ sufficiently small has three 2π -periodic orbits, $\gamma_i = (r_i(\theta, \varepsilon), Z_{1_i}(\theta, \varepsilon))$, such that $(r_i(0, \varepsilon), Z_{1_i}(0, \varepsilon)) \rightarrow \mathbf{S}_i$ as $\varepsilon \rightarrow 0$, with $i = 1, 2, 3$.

Moreover, the periodic solution γ_1 is stable if the real parts of the following eigenvalues of the Jacobian matrix $D_{\mathbf{y}} \mathbf{g}(\mathbf{S}_1)$

$$\lambda_1 = -\frac{a\omega^2 - c}{\omega^3},$$

and

$$\lambda_2 = \frac{4\omega^2 b_1 a - 2\omega^4 a_3 a - \omega^2 c a_3 - c b_1}{3\omega^3 (-\omega^2 a_3 + b_1)},$$

are strictly negative.

The periodic orbits γ_2 and γ_3 are stable if the real parts of the following eigenvalues of Jacobian matrices $D_y \mathbf{g}(\mathbf{S}_2)$ and $D_y \mathbf{g}(\mathbf{S}_3)$

$$\hat{\lambda}_1 = \frac{3\omega^2 ca_3 - 3cb_1 + \sqrt{25\omega^4 c^2 a_3^2 - 34\omega^2 c^2 a_3 b_1 - 23c^2 b_1^2 - 128\omega^4 a_3 cb_1 a + 32\omega^6 a_3^2 ca + 128c\omega^2 b_1^2 a}}{4(\omega^2 a_3 - 2b_1)\omega^3},$$

and

$$\hat{\lambda}_2 = \frac{3\omega^2 ca_3 - 3cb_1 - \sqrt{25\omega^4 c^2 a_3^2 - 34\omega^2 c^2 a_3 b_1 - 23c^2 b_1^2 - 128\omega^4 a_3 cb_1 a + 32\omega^6 a_3^2 ca + 128c\omega^2 b_1^2 a}}{4(\omega^2 a_3 - 2b_1)\omega^3},$$

are strictly negative.

This completes the proof of Theorem 2.1. $\square$

References

- [1] Bogolyubov, N. N. (1945). On some statistical methods in mathematical physics. Izdat. Akad. Nauk Ukr. SSR, Kiev.
- [2] Bogolyubov, N. N., & Krylov, N. (1934). The application of methods of nonlinear mechanics in the theory of stationary oscillations. Publication 8 of the Ukrainian Academy of Sciences, Kiev.
- [3] Buică, A., & Llibre, J. (2004). Averaging methods for finding periodic orbits via Brouwer degree. *Bulletin des Sciences Mathématiques*, 128, 7–22.
- [4] Davey, A. (1961). Boundary-layer flow at a saddle point of attachment. *Journal of Fluid Mechanics*, 10(4), 593–610.
- [5] Davey, A., & Schofield, D. (1967). Three-dimensional flow near a two-dimensional stagnation point. *Journal of Fluid Mechanics*, 28(1), 149–151.
- [6] Diab, Z., Guirao, J. L., & Llibre, J. (2025). Zero-Hopf bifurcation of a 5D hyperchaotic quadratic polynomial differential systems. *Mathematics and Computers in Simulation*, 238, 383–387.
- [7] Diab, Z., Guirao, J. L., & Vera, J. A. (2021). Zero-Hopf bifurcation in a generalized Genesio differential equation. *Mathematics*, 9(4), 354.
- [8] Fatou, P. (1928). Sur le mouvement d'un système soumis à des forces à courte période. *Bulletin de la Société Mathématique de France*, 56, 98–139.
- [9] Hiemenz, K. (1911). Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder. Göttingen dissertation. *Dingler's polytech. J.*, 326, 311.
- [10] Homann, F. (1936). Der Einfluss grosser Zähigkeit bei der Strömung um den Zylinder und um die Kugel. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 16(3), 153–164.
- [11] Howarth, L. (1951). XXV. The boundary layer in three dimensional flow.—Part I. Derivation of the equations for flow along a general curved surface. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 42(326), 239–243.
- [12] Khan, M., Sarfraz, M., Ahmed, J., Ahmad, L., & Fetecau, C. (2020). Non-axisymmetric Homann stagnation-point flow of Walter's B nanofluid over a cylindrical disk. *Applied Mathematics and Mechanics*, 41(5), 725–740.
- [13] Llibre, J. (2003). Averaging theory and limit cycles for quadratic systems. *Rad. Mat.*, 11, 215–228.
- [14] Llibre, J., & Pérez-Chavela, E. (2014). ZERO-HOPF BIFURCATION FOR A CLASS OF LORENZ-TYPE SYSTEMS. *Discrete & Continuous Dynamical Systems-Series B*, 19(6), 1731.
- [15] Merkin, J. H., Pop, I., Lok, Y. Y., & Grosan, T. (2021). Similarity solutions for the boundary layer flow and heat transfer of viscous fluids, nanofluids, porous media, and micropolar fluids. Academic Press.
- [16] Sanders, J. A., Verhulst, F., & Murdock, J. A. (2007). Averaging methods in nonlinear dynamical systems (Vol. 59, pp. 1–19). New York: Springer.
- [17] Uribe, M., & Martínez, E. (2021). Zero-Hopf bifurcation in the generalized Hiemenz equation. *Mathematical Methods in the Applied Sciences*, 44(11), 8565.
- [18] Wang, C. Y. (2008). Similarity stagnation point solutions of the Navier–Stokes equations—review and extension. *European Journal of Mechanics-B/Fluids*, 27(6), 678–683.
- [19] Wang, C. Y. (2008). Stagnation flow towards a shrinking sheet. *International Journal of Non-Linear Mechanics*, 43(5), 377–382.
- [20] Weidman, P. D. (2012). Obliquely-intersecting Hiemenz flows: a new interpretation of Howarth stagnation-point flows. *Fluid Dynamics Research*, 44(6), 065509.
- [21] Weidman, P., & Turner, M. R. (2017). Stagnation-point flows with stretching surfaces: A unified formulation and new results. *European Journal of Mechanics-B/Fluids*, 61, 144–153.