

Analysis of Multi-Dimensional Nonlinear Mathematical Physics Equations Traveling Wave Solutions and Initial Edge Value Problems

Tao Wang^a, Gan Yang^a, Yang Jin^a, Pengtao Chen^a, Xiuping Liu^a

^aHighway School, Chang'an University, Xi'an, Shaanxi, 710064, China

Abstract. Exploring the traveling wave solutions of mathematical physics equations and their boundary problems is an important work in mathematical physics research. In this paper, three types of representative multidimensional nonlinear equations are selected and the traveling wave solutions of the equations under different initial and side value conditions are studied. From the perspective of dynamical system theory, the treatment of parameter transformation is used to transform them into planar dynamical systems, and then the phase diagram of the planar system under the corresponding parameter conditions is obtained by using the dynamical system branching method, from which the existence of isolated wave solutions, isolated sharp waves and torsion waves of the perturbation equations and coupled field equations are obtained. When $\sigma f\left(\frac{c+\mu}{\sigma}\right) > 0$, the Camassa-Holm system has 11 branches of traveling wave solutions. When $\sigma f\left(\frac{c+\mu}{\sigma}\right) \leq 0$, the Camassa-Holm system has 6 branches of traveling wave solutions. The Boussinesq equation has a unique non-extendable solution defined on the interval under the first boundary condition and a generalized solution under the second boundary condition, while the Klein-Gordon equation has three singular points in different branches of the traveling wave solution. The present study is important for further understanding the dynamical properties of nonlinear systems.

1. Introduction

In today's linear science theory is becoming more and more perfect, the study of nonlinear science has attracted great attention. Because in people's real life as well as in nature, many phenomena and problems can no longer be described by linear theories [1, 2, 3]. Thus, the study of nonlinear science has become a common focus of attention. Since the 1980s and 1990s, the research in the field of nonlinear science has made a rapid development. Among them, the development of theories of isolators, fractals, and chaos are typical of them [4, 5, 6]. In particular, isolator theory plays an extremely important role in nonlinear science. In general, the study of the properties related to isolators usually translates into the study of the solutions of their corresponding nonlinear mathematical physics equations. In recent years, the study of traveling wave solutions of nonlinear mathematical physics equations has attracted much attention from scholars. The

Corresponding author: Gan Yang, mail address: ygchd@chd.edu.cn

Received: 15 February 2026; Accepted: 10 July 2026; Published: 5 August 2026

Keywords. Traveling wave solutions; representative multidimensional nonlinear equations; dynamical system theory; Camassa-Holm system

2010 *Mathematics Subject Classification.* 35C07, 35Q53, 37K05

Cited this article as: Wang, T., Yang, G., Jin, Y., Chen, P. & Liu, X. (2024). Analysis of multi-dimensional nonlinear mathematical physics equations traveling wave solutions and initial edge value problems. *Engineering and its Applications*, 3(1), 1–13.

© 2026 Engineering Journals. Published by Engineering Journals.

isolated wave solution, which has attracted considerable attention, is actually one of the cases of traveling wave solutions, and the study of it has become an important topic in the study of isolators [7, 8, 9].

The wave problem is an important research area in nonlinear mathematical equations, and its theory of thought, research methods and practical applications involve many disciplines such as mathematics, physics, chemistry and communication. The famous de Broglie matter wave theory, superconductivity, and turbulence are closely related to it [10, 11]. Generally, an equation can be called a fluctuation equation if there exists a solution that portrays the fluctuation behavior. Usually fluctuation equations are a class of spatio-temporal systems consisting of time and space variables. At the early stage of research, people often give idealized and simplistic assumptions for a practical fluctuation problem due to the limitation of understanding in order to obtain linear models with relatively low difficulty, such as heat conduction equation, string vibration equation, etc. With the advancement of science and technology and the continuous improvement of human cognitive level, especially the rise of nonlinear science, it has been found that linear models with overly demanding assumptions can no longer satisfactorily portray an objectively existing complex fluctuation phenomenon, and instead, nonlinear models, i.e., nonlinear fluctuation equations, also known as nonlinear evolution equations, should be used instead [12, 13]. Nonlinear fluctuation equations include nonlinear ordinary differential equations, nonlinear partial differential equations, and nonlinear mappings.

In recent years, some mathematical physics equations have become of increasing interest. In the literature [14], a linear system of mathematical physics equations in Hilbert space was studied, and the non-degenerate system was verified in terms of quadratic integration, Hamiltonian property and complete productability of the system. In the literature [15], for fractional differential equations, a fractional complex transformation is used to transform them into ordinary differential equations and the method is applied to three classes of mathematical physics equations, and the results show some feasibility. The literature [16] analyzed the invariance of stochastic differential equations by taking stochastic differential homogeneous embryos as an example, and studied the traveling wave solutions of mathematical physics equations by studying the different forms of the equations determined by the symmetry of stochastic Lie points. In the literature [17], the Ablowitz-Kaup-Newell-Segur equation was analyzed based on the Hirota bilinear method to convert it into the standard and nonlocal nonlinear Schrödinger equation to find isolated solutions of the mathematical physics equation.

In the literature [18], the wave structure of solutions in different spatial dimensions was explored by a point-by-point estimation method for the system space of compressible Navier-Stokes equations. The literature [19] analyzed in depth the positive solutions of equations with double-weighted critical exponents based on Hardy-Sobolev, Sobolev and Hénon-Sobolev exponents. In the literature [20], the flow equations were established based on a one-dimensional inviscid compressible flow process, and the equations were formulated as an exterior differential system using an isotropic symmetry analysis method. In the literature [21], the existence of traveling wave solutions to the fractional Kirchhoff problem was demonstrated by combining the non-Nehari flow form method and the new equation solution method, and the energy of the ground state solution as well as the convergence properties were explored.

In this paper, three types of multidimensional nonlinear mathematical physical equations are studied with respect to the traveling wave solutions and the dynamical forms under the initial edge value condition. Based on the asymptotic regression analysis of the two-dimensional rotating gravity water wave control equation, the rotating two-component Camassa-Holm equation is established, transformed into a planar system by variable substitution, and further transformed into a Hamiltonian system based on the singular line. Based on the equilibrium points and phase trajectories of the Hamiltonian system, the traveling wave solutions of different branches are derived through parametric discussions. For the “bad” Boussinesq type equation, the solution of the initial value problem is considered as a “flow” from the initial point, based on the topological invariance of the isometric homogeneous space, the Galrekin approximation and the successive extensions of the solution. The combination of topological invariance of isometric isomorphic space, Galrekin approximation and successive extension of the solution, proves the existence of traveling wave solutions. For the Klein-Gordon equation, the Hamiltonian system and its Hamiltonian function are obtained by the traveling wave transform, the saddle point and the center point are discussed under different parameter conditions, and the parametric expressions of the traveling wave solution are solved

by the phase trajectory.

2. Traveling Wave Solutions and Initial Values of the Rotating Two-Component Camassa-Holm Equation

2.1. Rotating two-component Camassa-Holm equation

2.1.1. Establishment of mathematical physical equations

Based on the asymptotic regression analysis of the two-dimensional rotating gravity water wave control equation, the researchers derived the rotating two-component Camassa-Holm system, which simulates equatorial water waves under the Coriolis force in a rotating fluid:

$$\begin{cases} u_t - u_{xx} - Au_x + 3uu_x = \sigma(2u_x u_{xx} + uu_{xx}) - \mu u_{xxx} - (1 - 2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x \\ \rho_t + (\rho u)_x = 0 \end{cases} \quad (1)$$

Where $u(x, t)$ is the fluid velocity in the x -direction, $\rho(x, t)$ denotes the height of the free surface after equilibrium, parameter $A > 0$ denotes the linear bottom shear flow, σ is an equilibrium-dependent parameter, μ is a dimensionless parameter, and Ω is the constant rotational speed of the Earth.

2.1.2. Planar system conversion

To study the traveling wave solution of equation (1), the transformations $u(x, t) = \varphi(\xi)$, $\rho(x, t) = \psi(\xi)$, $\xi = x - ct$ are made such that we have:

$$\begin{cases} -c\varphi' + c\varphi''' - A\varphi' + 3\varphi\varphi' = \sigma(2\varphi'\varphi'' + \varphi\varphi'') - \mu\varphi''' - (1 - 2\Omega A)\psi\psi' + 2\Omega\psi(\psi\varphi)' \\ -c\psi' + (\psi\varphi)' = 0 \end{cases} \quad (2)$$

Direct integration of (2) once gives:

$$\begin{cases} G - (c + A)\varphi + c\varphi'' + \frac{3}{2}\varphi^2 = \frac{\sigma}{2}(\varphi')^2 + \sigma\varphi\varphi'' - \mu\varphi'' - \left(\frac{1}{2} - \Omega A - \Omega c\right)\psi^2 \\ -c\psi + \psi\varphi = g \end{cases} \quad (3)$$

Where both G and g are integration constants. Obviously, from (3) we get that:

$$\psi = \frac{g}{\varphi - c} \quad (4)$$

Thus there are:

$$(\sigma\varphi - c - \mu)(\varphi - c)^2\varphi'' = -\frac{\sigma}{2}(\varphi - c)^2(\varphi')^2 + (\varphi - c)^2\left(\frac{3}{2}\varphi^2 - (c + A)\varphi + G\right) + \left(\frac{1}{2} - \Omega(A + c)\right)g^2 \quad (5)$$

Let $\frac{d\varphi}{d\xi} = y$, a planar system is obtained:

$$\begin{cases} \frac{d\varphi}{d\xi} = y \\ \frac{dy}{d\xi} = \frac{-\frac{\sigma}{2}(\varphi - c)^2y^2 + (\varphi - c)^2\left(\frac{3}{2}\varphi^2 - (c + A)\varphi + G\right) + \left(\frac{1}{2} - \Omega(A + c)\right)g^2}{(\sigma\varphi - c - \mu)(\varphi - c)^2} \end{cases} \quad (6)$$

The system (6) has two odd lines with Hamiltonian functions as:

$$H(\varphi, y) = \frac{1}{2}(\sigma\varphi - c - \mu)y^2 - \frac{1}{2}\varphi^3 + \frac{1}{2}(c + A)\varphi^2 - G\varphi + \frac{\left(\frac{1}{2} - \Omega(A + c)\right)g^2}{\varphi - c} = h \quad (7)$$

Let $d\xi = (\sigma\varphi - c - \mu)(\varphi - c)^2 d\tau$, the system (6) transforms into the corresponding Hamiltonian system:

$$\begin{cases} \frac{d\varphi}{d\tau} = (\sigma\varphi - c - \mu)(\varphi - c)^2 y \\ \frac{dy}{d\tau} = -\frac{\sigma}{2}(\varphi - c)^2 y^2 + (\varphi - c)^2 \left(\frac{3}{2}\varphi^2 - (c + A)\varphi + G \right) + \left(\frac{1}{2} - \Omega(A + c) \right) g^2 \end{cases} \quad (8)$$

Notice that the system (8) and the system (6) have the same Hamiltonian function. Therefore, the phase diagram of system (6) can be obtained by studying the phase diagram of system (8).

In order to study the equilibrium point of the system (8) and its properties, it is useful to set:

$$f(\varphi) = (\varphi - c)^2 \left(\frac{3}{2}\varphi^2 - (c + A)\varphi + G \right) + \frac{1}{2} - \Omega(A + c) \quad (9)$$

From the above equation, it is easy to see that:

$$f'(\varphi) = (\varphi - c)(6\varphi^2 - 3(A + 2c)\varphi + c(A + c) + 2G) \quad (10)$$

$$f''(\varphi) = 18\varphi^2 - 6(A + 4c)\varphi + c(4A + 7c) + 2G \quad (11)$$

If $\Delta = 9A^2 + 12Ac + 12c^2 - 48G > 0$, then there are three zeros as follows:

$$\hat{\varphi}_0 = c, \quad \hat{\varphi}_{\pm} = \frac{1}{12} (3(A + 2c) \pm \sqrt{\Delta}) \quad (12)$$

And easy to get $f(c) = \frac{1}{2} - \Omega(A + c)$, $f'(c) = 0$, and $f''(c) = c^2 - 2Ac + 2G$.

On the φ -axis, the system (8) has at most four equilibrium points, denoted by $S_i(\varphi_i, 0)$, $i = 1, 2, 3, 4$. On the odd line l_1 , if $g \neq 0$ and $\Omega \neq \frac{1}{2(A + c)}$, then the system (8) does not have equilibrium points.

In fact, when $g = 0$, the system (8) is meaningless. And on the odd line l_2 , if $\sigma f\left(\frac{c + \mu}{\sigma}\right) > 0$, the system (8) exists in two equilibrium points $S_{\pm}\left(\frac{c + \mu}{\sigma}, y_{\pm}\right)$. Among them, there is

$$y_{\pm} = \pm \sqrt{\frac{2f\left(\frac{c + \mu}{\sigma}\right)}{\sigma\left(\frac{c + \mu}{\sigma} - c\right)^2}}.$$

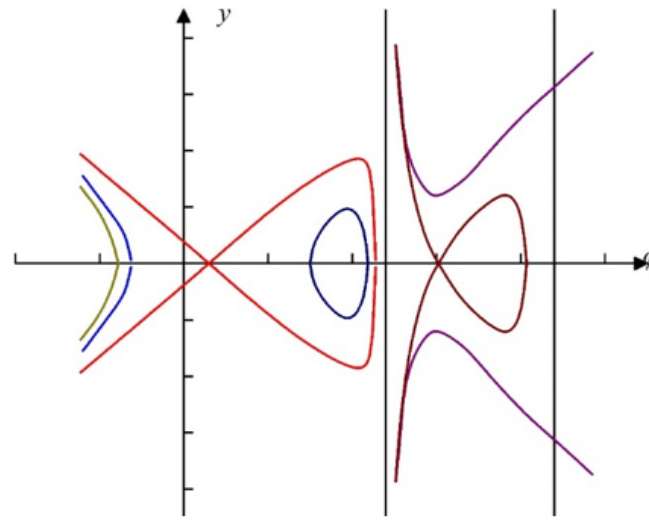
2.2. Branching phase diagram and initial edge value analysis of the traveling wave solution

2.2.1. Branching phase diagrams of traveling wave solutions under different conditions

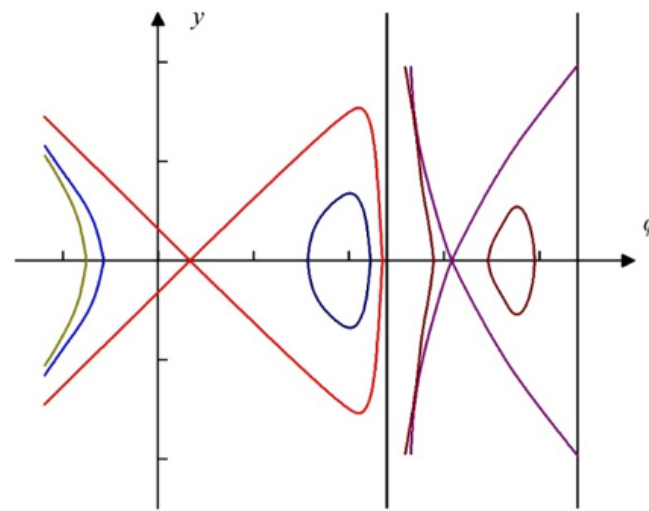
Let $h_1 = H(\varphi_i, 0)$, $i = 1, 2, 3, 4$ and $h_s = H\left(\frac{c + \mu}{\sigma}, y_{\pm}\right)$, where $h_s = H(\varphi, y)$ is given above. Then, we can obtain the phase diagram of the system (6).

$$1. \quad \Omega < \frac{1}{2(A + c)}, \quad \sigma > 0$$

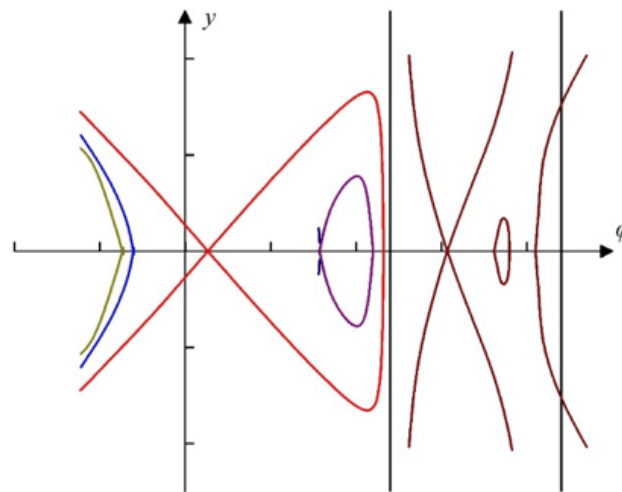
When $\varphi_4 < \frac{c + \mu}{\sigma}$, the phase diagrams of the system corresponding to different parameter relations are shown in Figure 1. where the condition of Figure 1(a) is $h_1 < h_2 < h_3 < h_4$. the condition of Figure 1(b) is $h_1 < h_2 < h_3 = h_s < h_4$. the condition of Figure 1(c) is $h_1 < h_2 < h_3 < h_s < h_4$.



(a) $h_1 < h_2 < h_3 < h_4$



(b) $h_1 < h_2 < h_3 = h_s < h_4$



(c) $h_1 < h_2 < h_3 < h_s < h_4$

Figure 1: Branching phase diagram for the first parameter condition

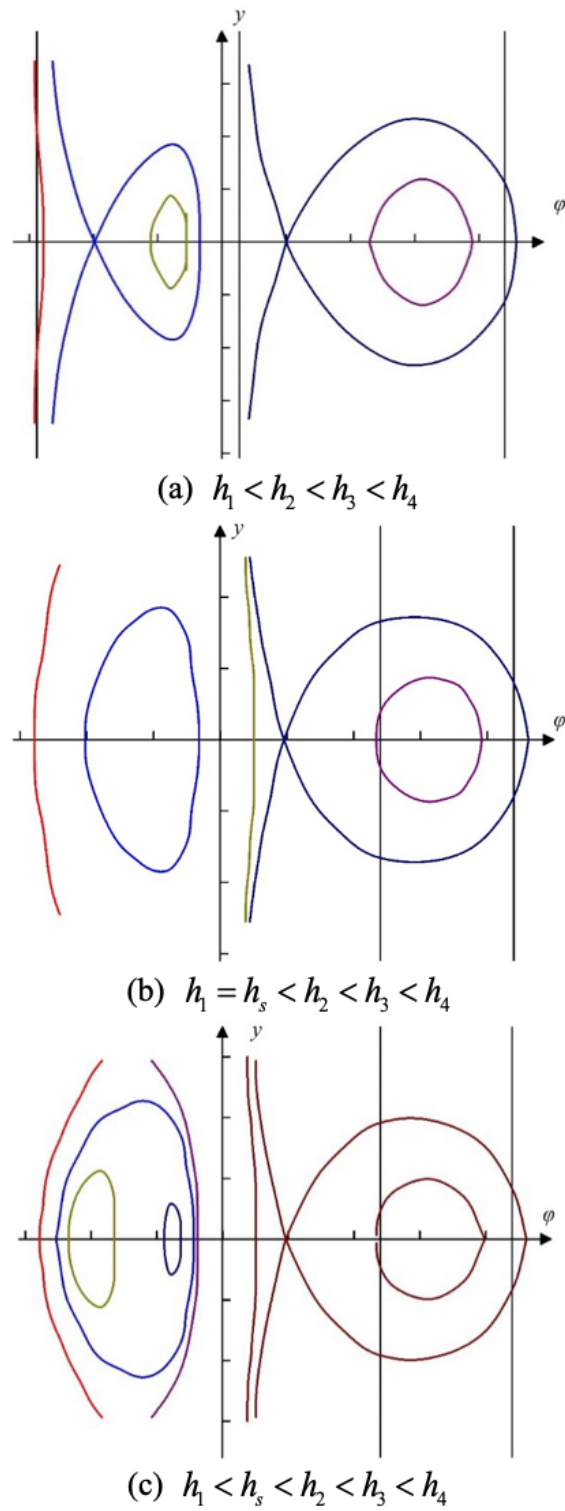


Figure 2: Branching phase diagram for the second parameter condition

$$2. \Omega < \frac{1}{2(A+c)}, \sigma < 0$$

When $\frac{c+\mu}{\sigma} < \varphi_1$, the phase diagrams of the system corresponding to different parameter relations are shown in Figure 2. where the condition of Figure 2(a) is $h_1 < h_2 < h_3 < h_4$. the condition of Figure 2(b) is $h_1 = h_s < h_2 < h_3 < h_4$. the condition of Figure 2(c) is $h_1 < h_s < h_2 < h_3 < h_4$.

$$3. \Omega < \frac{1}{2(A+c)}, \sigma > 0, \varphi_1 < c < \varphi_2$$

When $\varphi_4 < \frac{c+\mu}{\sigma}$, the phase diagrams of the system corresponding to different parameter relations are shown in Figure 3. Where the condition of Fig. 3(a) is $h_s < h_3 < h_2 < h_4 < h_1$. the condition of Fig. 3(b) is $h_s = h_3 < h_2 < h_4 < h_1$. the condition of Fig. 3(c) is $h_3 < h_s < h_2 < h_4 < h_1$.

2.2.2. Initial edge value problem for standing wave solutions

For the sake of illustration, let $sn(\cdot, \cdot)$ be the Jacobi elliptic function and $\Pi(\cdot, \cdot, \cdot)$ be the third class elliptic integral. Furthermore according to Eq. (7), for a fixed h , we have:

$$y^2 = \frac{\varphi^4 - (A+2c)\varphi^3 + (c^2 + Ac + 2G)\varphi^2 + (2h - 2cG)\varphi - (2ch + 1 - 2\Omega(A+c))}{(\sigma\varphi - c - \mu)(\varphi - c)} \triangleq \frac{T(\varphi)}{(\sigma\varphi - c - \mu)(\varphi - c)} \quad (13)$$

We give the main results about isolated waves, isolated sharp waves and torsional waves in the following exposition.

1. Co-habitation orbit through saddle point $S_1(\varphi_1, 0)$

In Figure 1(a-b), there exists an isolated wave of equation (1) with an implicit expression:

$$\begin{aligned} & \left(c + \theta_{11} + \frac{\theta_{12}}{c - \varphi_1} \right) sn^{-1} \sqrt{\frac{(\varphi_{12} - c)(\varphi_{11} - \varphi)}{(\varphi_{12} - \varphi_{11})(c - \varphi)}} + (\varphi_{11} - c) \Pi \left(\sin^{-1} \sqrt{\frac{(\varphi_{12} - c)(\varphi_{11} - \varphi)}{(\varphi_{12} - \varphi_{11})(c - \varphi)}}, \alpha_{11}^2, k_1 \right) \\ & + \frac{\theta_{12}(c - \varphi_{11})}{(\varphi_{11} - \varphi_1)(c - \varphi_1)} \Pi \left(\sin^{-1} \sqrt{\frac{(\varphi_{12} - c)(\varphi_{11} - \varphi)}{(\varphi_{12} - \varphi_{11})(c - \varphi)}}, \alpha_{11}^2, k_1 \right) = \frac{|\xi|}{g_1 \sqrt{\sigma}} \end{aligned} \quad (14)$$

Where $g_1 = \frac{2}{\sqrt{(\varphi_{12} - c) \left(\frac{c+\mu}{\sigma} - \varphi_{11} \right)}}$, φ_{1i} , $i = 1, 2$, is obtained by making $h = h_1$ and having:

$$T(\varphi) = (\varphi - \varphi_{12})(\varphi - \varphi_{11})(\varphi - \varphi_1)^2 \quad (15)$$

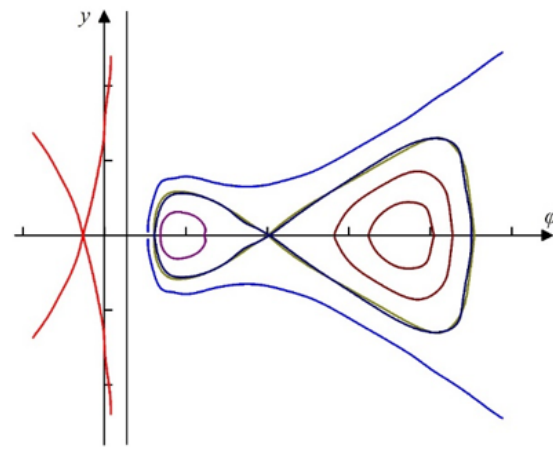
In Figure 1(c), there exists an isolated wave of equation (1) with an implicit expression:

$$\frac{2\sqrt{\Phi(\varphi)} + 2\varphi - (\varphi_{11} + \varphi_{12})}{\varphi_{11} - \varphi_{12}} = e^{\frac{|\xi|}{g_1}} \quad (16)$$

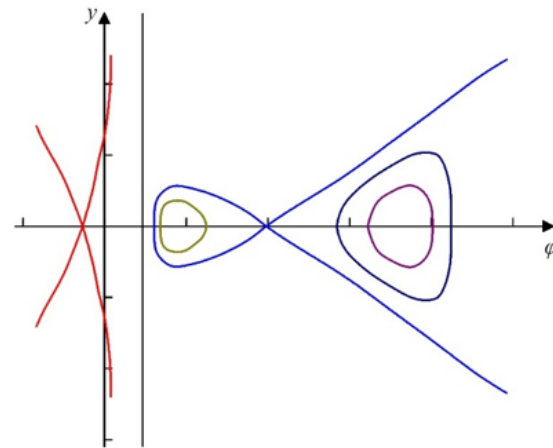
Where $\Phi(\varphi) = (\varphi_{11} - \varphi)(\varphi_{12} - \varphi)$, φ_{1i} , $i = 1, 2$ is obtained by making $h = h_1$ and having:

$$T(\varphi) = (\varphi - \varphi_{12})(\varphi - \varphi_{11})(\varphi - \varphi_1)^2 \quad (17)$$

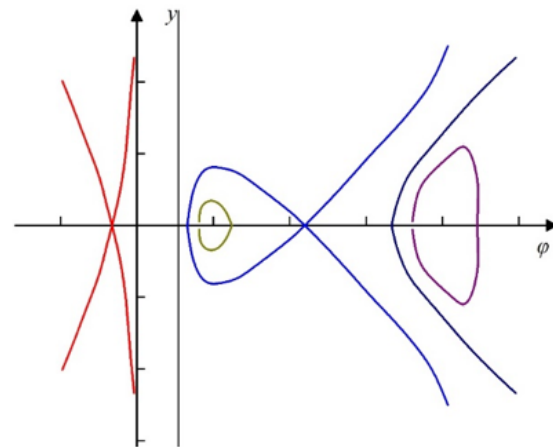
2. Co-habitation orbit through saddle point $S_2(\varphi_2, 0)$



(a) $h_s < h_3 < h_2 < h_4 < h_1$



(b) $h_s = h_3 < h_2 < h_4 < h_1$



(c) $h_3 < h_s < h_2 < h_4 < h_1$

Figure 3: Branching phase diagram for the third parameter condition

In Figure 2(a), there exists an isolated wave of equation (1) with the implicit expression:

$$\begin{aligned} & \left(-\varphi_{22} + \theta_{21} + \frac{\theta_{22}}{\varphi_2 - \varphi_{22}} \right) sn^{-1} \sqrt{\frac{\left(\varphi_{22} - \frac{c + \mu}{\sigma} \right) (\varphi - \varphi_{21})}{\left(\frac{c + \mu}{\sigma} - \varphi_{21} \right) (\varphi_{22} - \varphi)}} + (\varphi_{22} - \varphi_{21}) \Pi \left(\sin^{-1} \sqrt{\frac{\left(\varphi_{22} - \frac{c + \mu}{\sigma} \right) (\varphi - \varphi_{21})}{\left(\frac{c + \mu}{\sigma} - \varphi_{21} \right) (\varphi_{22} - \varphi)}}, \alpha_{21}^2, k_2 \right) \\ & + \frac{\theta_{22}(\varphi_{21} - \varphi_{22})}{(\varphi_2 - \varphi_{21})(\varphi_2 - \varphi_{22})} \Pi \left(\sin^{-1} \sqrt{\frac{\left(\varphi_{22} - \frac{c + \mu}{\sigma} \right) (\varphi - \varphi_{21})}{\left(\frac{c + \mu}{\sigma} - \varphi_{21} \right) (\varphi_{22} - \varphi)}}, \alpha_{21}^2, k_2 \right) = \frac{|\xi|}{g_2 \sqrt{-\sigma}} \end{aligned} \quad (18)$$

Where $g_2 = \frac{2}{\sqrt{\left(\varphi_{22} - \frac{c + \mu}{\sigma} \right) (c - \varphi_{21})}}$, φ_{2i} , $i = 1, 2$ is obtained by making $h = h_2$ and having:

$$T(\varphi) = (\varphi - \varphi_{22})(\varphi - \varphi_2)^2(\varphi - \varphi_{21}) \quad (19)$$

In Figure 2(b-c), there exists an isolated wave of equation (1) with an implicit expression:

$$\frac{\varphi_{21} - \varphi_{22}}{2\sqrt{\Phi(\varphi)} + 2\varphi - (\varphi_{21} + \varphi_{22})} \times \left(\frac{(\varphi_{22} - \varphi_{21})(\varphi_{21} - \varphi_2)}{2\sqrt{\Phi(\varphi)\Phi(\varphi_2)} + 2\Phi(\varphi_2) + (2\varphi_2 - \varphi_{21} - \varphi_{22})(\varphi - \varphi_2)} \right)^{\frac{c - \varphi_1}{\sqrt{\Phi(\varphi_2)}}} = e^{\frac{|\xi|}{\sqrt{\sigma}}} \quad (20)$$

Where $\Phi(\varphi) = (\varphi_{22} - \varphi)(\varphi_{21} - \varphi)$, φ_{2i} , $i = 1, 2$ is obtained by making $h = h_2$ and having:

$$T(\varphi) = (\varphi - \varphi_{22})(\varphi - \varphi_{21})(\varphi - \varphi_2)^2 \quad (21)$$

3. Co-habitation orbit through saddle point $S_3(\varphi_3, 0)$

In Figure 3(a-b), there exists an isolated wave of equation (1) with the implicit expression:

$$\begin{aligned} & \left(-\varphi_{22} + \theta_{21} + \frac{\theta_{22}}{\varphi_2 - \varphi_{22}} \right) sn^{-1} \sqrt{\frac{(\varphi_{22} - c)(\varphi_{21} - \varphi)}{\left(\frac{c + \mu}{\sigma} - \varphi_{21} \right) (\varphi_{22} - \varphi)}} + (\varphi_{22} - \varphi_{21}) \Pi \left(\sin^{-1} \sqrt{\frac{(\varphi_{22} - c)(\varphi_{21} - \varphi)}{\left(\frac{c + \mu}{\sigma} - \varphi_{21} \right) (\varphi_{22} - \varphi)}}, \alpha_{21}^2, k_2 \right) \\ & + \frac{\theta_{22}(\varphi_{22} - \varphi_{21})}{(\varphi_{21} - \varphi_2)(\varphi_{22} - \varphi_2)} \Pi \left(\sin^{-1} \sqrt{\frac{(\varphi_{22} - c)(\varphi_{21} - \varphi)}{\left(\frac{c + \mu}{\sigma} - \varphi_{21} \right) (\varphi_{22} - \varphi)}}, \alpha_{21}^2, k_2 \right) = \frac{|\xi|}{g_2 \sqrt{\sigma}} \end{aligned} \quad (22)$$

Where $g_2 = \frac{2}{\sqrt{\left(\varphi_{22} - \frac{c + \mu}{\sigma} \right) (c - \varphi_{21})}}$, and φ_{2i} , $i = 1, 2$ is obtained by making $h = h_2$ and having:

$$T(\varphi) = (\varphi - \varphi_{22})(\varphi - \varphi_{21})(\varphi - \varphi_2)^2 \quad (23)$$

In Figure 3(b-c), there exists an isolated wave of equation (1) with an implicit expression:

$$\frac{2\sqrt{\Phi(\varphi)} + 2\varphi - (c + \varphi_{22})}{\varphi_{22} - c} \times \left(\frac{(c - \varphi_{22})(\varphi_{22} - \varphi_2)}{2\sqrt{\Phi(\varphi)\Phi(\varphi_2)} + 2\Phi(\varphi_2) + (2\varphi_2 - \varphi_{22} - c)(\varphi - \varphi_2)} \right)^{\frac{\varphi_2 - c}{\sqrt{\Phi(\varphi_2)}}} = e^{\frac{|\xi|}{\sqrt{\sigma}}} \quad (24)$$

Where $\Phi(\varphi) = (c - \varphi)(\varphi_n - \varphi)$, φ_{2i} , $i = 1, 2$ is obtained by making $h = h_2$ and having:

$$T(\varphi) = (\varphi - \varphi_{22})(\varphi - \varphi_2)^2(\varphi - \varphi_{21}) \quad (25)$$

3. Traveling Wave Solutions and Extensions of Boussinesq-Type Equations

In this section, we devote ourselves to discussing the “bad” Boussinesq-type equations:

$$u_{tt} - u_{xx} - bu_{xxx} = \sigma(u)_{xx}, \quad 0 < x < 1, t > 0 \quad (26)$$

When it has the following boundary conditions:

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad t > 0 \quad (27)$$

Or when its boundary conditions:

$$u_x(0, t) = u_x(1, t) = u_{xxx}(0, t) = u_{xxx}(1, t) = 0, \quad t > 0 \quad (28)$$

And an initial condition of:

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \phi(x), \quad 0 \leq x \leq 1 \quad (29)$$

Study the existence and solution of the Blowup problem for the traveling wave solution of its initial edge value problem, where $b > 0$ is a constant and $\sigma(s)$ is a given function.

In this paper, we will show that the first boundary condition problem consisting of Eq. (26)(27)(29) and the second boundary condition problem consisting of Eq. (26)(28)(29) have local traveling wave solutions as long as $\sigma \in C^2(\mathbb{R})$, local Lipschitz continuity of $\sigma''(s)$, $\varphi \in H^4$, $\phi \in L^2$, and compatibility boundary conditions are satisfied. And under further restriction conditions for the initial function, the above traveling wave solution will be Blowup at finite moments.

In this paper, we use the idea of treating the solution of the initial value problem of an equation as a “flow” from the initial point in the theory of ordinary differential equations, and use the topological invariance of isometric homogeneous spaces, the Galerkin approximation and the method of successive extensions of the solution to investigate the existence of the initial side value problem and the solution of the “bad” Boussinesq-type equation.

3.1. Boussinesq type equation

Waves are one of the most common natural phenomena in oceans and estuaries. The motion law of waves is important for the study of shore and beach evolution, dike and other building forces, channel and harbor design, sediment and pollutant transport and dispersion in the water and other issues. With the rapid development of the economy and the demand for marine energy, more and more coastal and marine projects are being implemented or will be implemented, the scale of the project is expanding and the implementation requirements are increasing, therefore, it is increasingly important and urgent to study the law of motion of wave field in the sea in a profound and detailed way. Especially in offshore projects, waves are an important influence factor that cannot be ignored.

The mathematical model based on Boussinesq equation can describe many deformations of waves in gently varying topography and currents of constant water depth, such as nonlinearity, dispersion, amplitude dispersion, linear and nonlinear shallowing, multidirectional propagation, bypassing, refraction caused by topography, refraction under the influence of currents, harmonics, etc. Moreover, it does not separate incident and reflected waves distinctly, and can consider the combined effects of topography, buildings and nonlinear. The mathematical model is widely used to solve the wave field in the nearshore complex engineering region. Although the mathematical form of this equation is a shallow water wave equation, it has been improved to extend its applicability to deep water. With the improvement of the performance of the equation and the continuous improvement of the solution method, the model has been greatly improved for the simulation of various physical phenomena mentioned above.

3.2. First boundary condition

3.2.1. Existence of traveling wave solutions

Appointment:

$$Q_T = (0, 1) \times (0, T), \quad L_2 = L_2(0, 1), \quad H^k = H^k(0, 1) \quad (k = 1, 2, \dots), \quad \|\cdot\| = \|\cdot\|_{L_2}, \quad |\cdot|_k = h_0 \quad (30)$$

The symbol $(\cdot, \cdot)$ denotes the dual product of a continuous linear generic function on an inner product linear space in L_2 -space with elements in that space.

$\forall f \in L_2, f = \sum_{k=1}^{\infty} f_k e_k$, where $f_k = (f, e_k)$, the sequence $\{e_k\}$ forms the orthogonal canonical base of L_2 .

Denote $f = (f_1, f_2, \dots, f_k, \dots)$, then

$$\|f\| = \left(\sum_{k=1}^{\infty} f_k^2 \right)^{1/2} = \|f\|_{l^2},$$

i.e. L_2 is isometric isomorphic to l^2 . Notation:

$$S = \{f \in H^4 : f(0) = f(1) = f''(0) = f''(1) = 0\} \quad (31)$$

Defining the parametrization $\|f\|_l = \left(\|f\|^2 + \|f^{(5)}\|^2 \right)^{1/2}$, then S is a Hilbert space. $S = S^+$, i.e., S is self-congruent, and $\{e_k\}$ is also an orthogonal basis of S .

For any f belonging to S , there are:

$$S = \{f \in l^2 : (\pi^4 f_1, \dots, (k\pi)^4 f_k, \dots) \in l^2\} \quad (32)$$

Let X^n and S^n be the n -dimensional subspaces generated by $\{e_1, e_2, \dots, e_n\}$ in L_2 and in S , respectively. $P_n : L_2 \rightarrow X^n$ is the orthogonal projection operator.

3.2.2. The initial edge value of the solution is extended

Let $\sigma \in C^2(\mathbb{R})$, σ'' locally Lipschitz continuous, and $\varphi \in S, \phi \in L_2$. Then there exists a generalized solution to the first boundary condition problem. Where $0 < T < T^0$ and $[0, T^0)$ is the maximum time existence interval of u . Furthermore, if:

$$\sup_{0 \leq t < T^0} \|u(t)\|_S < +\infty \quad (33)$$

then $T^0 = +\infty$.

Since the space X^n is isomorphic and homogeneous with $\mathbb{R}^n$, it is known from the standard theory of ordinary differential equations in $\mathbb{R}^n$ that there exists a unique non-extendable solution of the first boundary condition problem defined on the interval J_n , where J_n is the maximum time existence interval of $u^n(t)$.

Consider the initial value problem:

$$\begin{cases} (v^n)'(t) = g(v^n(t)), & t > 0, \\ v^n(0) = u^n(b_0), \quad (v^n)'(0) = (u^n)'(b_0) \end{cases} \quad (34)$$

Remember:

$$G(\delta) = \{(\tau, w) \in \mathbb{R} \times S : \text{dist}((\tau, w), G) < \delta\} \quad (35)$$

is the δ -neighborhood of the image G in $\mathbb{R} \times S$. Then:

$$\mu_n(\theta) = \{(\tau, w) \in \mathbb{R} \times S^n : \|\tau - b_0\| + \|w - u^n(b_0)\|_S < \theta\} \quad (36)$$

$$\mu_n(\theta) = \{(\tau, w) \in \mathbb{R} \times S^n : \|\tau - b_0\| + \|w - u^n(b_0)\|_S < \theta\} \quad (37)$$

are θ neighborhoods of $(b_0, u^n(b_0))$ in $\mathbb{R} \times S^n$ and $(b_0, u^n(b_0))$ in $\mathbb{R} \times S^n$, respectively. Obviously, a sufficient necessary condition for $(\tau, w) \in \mu_n(\theta)$ is $(\tau, w) \in \mu_n(\theta)$.

3.2.3. The successive extensions of the solution

In the bootstrap formulation, taking $d = 0$, $J = [0, 0]$, there exists a traveling wave solution $u(t) = \varphi$, $t \in J$ of the first boundary problem satisfying the condition $u(0) = \phi$. The corresponding function is the solution of the first boundary problem on J and satisfies the condition $u_t(0) = \phi$.

The problem has a solution $u(t) = \varphi^n$, $t \in [0, d_n]$ and satisfies the condition $u_t(0) = \phi$. It follows from the Lemma that $\varphi_t < \theta$ where θ is a positive constant that does not depend on n . Take a bounded open ball Q_1 in the space $\mathbb{R} \times S$ such that $(0, \varphi) \in Q_1$ and $\text{dist}(\partial Q_1, (0, \varphi)) \geq 3\theta$, then the conditions of the Lemma hold. Thus, there exists a positive constant b_1 and subcolumns that depend only on θ such that both $u^{1,n}(t)$ and $u(t)$ are extended to the interval $[0, b_1)$ and the bootstrap condition holds. In addition, u^n and d' are replaced by $u^{1,n}$ and b_1 , respectively.

3.3. Second boundary condition

3.3.1. Finding the traveling wave solution

Taking $e_0 = \frac{1}{2}$, $e_k = \cos k\pi x$ ($k = 1, 2, \dots$), then $\{e_k\}$ forms the orthogonal basis of L_2 and defines the Fourier transform:

$$L_2 \rightarrow l^2, \quad \forall f \in L_2, \quad \hat{f}(0) = 4(f, e_0) = f_0, \quad \hat{f}(k) = 2(f, e_k) = f_k, \quad k = 1, 2, \dots \quad (38)$$

Obviously, there are:

$$f_{-k} = f_k, \quad k = 0, 1, \dots, \quad f = \sum_{k=0}^{\infty} f_k e_k, \quad f^0 = (f_0, f_1, \dots, f_k, \dots) \quad (39)$$

Notes:

$$f = (f_0, f_1, \dots, f_k, \dots) \quad (40)$$

Then:

$$\|f\| = \left[\frac{1}{2} \left(\frac{f_0^2}{2} + \sum_{k=1}^{\infty} f_k^2 \right) \right]^{1/2} \quad (41)$$

Provisions:

$$D = \{f \in H^4 : f'(0) = f'(1) = f^{(3)}(0) = f^{(3)}(1) = 0\} \quad (42)$$

Then D is a Hilbert space, and $\{e_k\}$ is also an orthogonal basis of D .

3.3.2. Extensions of the solution

Based on the above convention, similar to the derivation of the first boundary condition, the extension of the traveling wave solution under the second boundary condition can be obtained.

Let $\sigma''(s)$ be locally Lipschitz continuous and $\varphi \in D$, $\phi \in L_2$, then there exists a generalized solution $u \in L_{\infty}([0, T]; D)$ to the second boundary condition problem. Where $0 < T < T^0$, $[0, T^0]$ is the maximum time existence interval of u . Furthermore, if:

$$\sup_{0 \leq t < T^0} \|u(t)\|_D < +\infty \quad (43)$$

Then $T^0 = +\infty$.

4. Periodic and Isolated Waves of the Klein-Gordon Equation

4.1. Establishment of the Klein-Gordon equation

The fundamental kinetic law of non-relativistic quantum mechanics (NRQM) is the Schrödinger equation. The Schrödinger equation in the coordinate representation can be obtained formally by applying the primary quantization assumption to the wave function characterizing the state of the microscopic particle based on the energy-momentum relation of the non-relativistic classical particle. Obviously, the Schrödinger equation is only invariant to the Galileo transformation. It should be emphasized that the Schrödinger equation for any potential field automatically implies the conservation of the number of particles, except in special cases where the wave function collapses, or where an expression with side conditions is required. This shows that NRQM only studies the spacetime motion of particles under the action of various potential fields and does not consider the transformation between different kinds of particles. Therefore, although it is not a classical theory, it is in accordance with the traditional concepts of mechanical theory.

In order to make the established quantum theory applicable to high-energy particles, it must be generalized and made compatible with the principle of special relativity. The specific way to find the equations of relativistic dynamics of microscopic particles is the primary quantization equation. This idea used to dominate the establishment of relativistic quantum mechanics (RQM) in the early development process of quantum theory. The Klein-Gordon equation and the Dirac equation were two products along this line.

The Klein-Gordon equation, originally, was intended as a generalization of the Schrödinger equation to the relativistic single-particle equation. At that time, in order to overcome the shortcomings of the Klein-Gordon equation and to obtain a correct RQM interpretation of the fine structure of the hydrogen atom, Dirac proposed a free single-particle equation, the Dirac equation. The equation derives the electron spin in a very natural way, and the solution of the central field also gives the correct answer for the fine structure of the hydrogen atom spectrum. In particular, the equation also predicts the existence of positron, a positively charged antiparticle, which is experimentally confirmed. In addition, the equations avoid the difficulties of probability density non-positive definiteness and the problem of second-order derivatives with respect to time. However, the Dirac equation, like the Klein-Gordon equation, still has negative energy solutions.

4.2. Characterization of differential systems

Based on the process in the above, the Klein-Gordon equation is established:

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0 \quad (44)$$

To (44) for the traveling wave transform to get $\xi = x - ct$, $\alpha u + \beta u^3 + (c^2 - 1)u'' = 0$, the following note $y_1 = u$, $y_2 = u'(\xi)$, then the Hamiltonian system is obtained:

$$\begin{cases} \frac{dy_1}{d\xi} = y_2 \\ \frac{dy_2}{d\xi} = my_1 + my_1^3 \end{cases} \quad (45)$$

Its Hamilton function is:

$$H(y_1, y_2) = \frac{2(-1 + c^2)y_2^2 + 2\alpha y_1^2 + \beta y_1^4}{4(-1 + c^2)} = h \quad (46)$$

Where h_1 is Hamilton's constant, the general solution of equation (45) is:

$$y_2 = \sqrt{\frac{-2\alpha y_1^2 - \beta y_1^4 - 4h + 4c^2h}{2(-1 + c^2)}} \quad (47)$$

And:

$$y_2 = -\sqrt{\frac{-2\alpha y_1^2 - \beta y_1^4 - 4h + 4c^2h}{2(-1 + c^2)}} \quad (48)$$

The Jacobi matrix of system (45) is:

$$A(y_1, y_2) = \begin{pmatrix} 0 & 1 \\ -\alpha - 3\beta y_1^2 & 0 \end{pmatrix}, \quad |A(y_1, y_2)| = \frac{\alpha + 3\beta y_1^2}{-1 + c^2} \quad (49)$$

4.3. Analysis of traveling wave solutions and initial edge value conditions

When $m < 0$ and $n > 0$, $\alpha\beta < 0$ and $c^2\alpha > \alpha$ the system (45) has three singularities $\left(\sqrt{-\frac{\alpha}{\beta}}, 0\right)$, $(0, 0)$ and $\left(-\sqrt{-\frac{\alpha}{\beta}}, 0\right)$. They are the saddle point, center and saddle point, respectively. The phase diagram of the system (45) is shown in Figure 4.

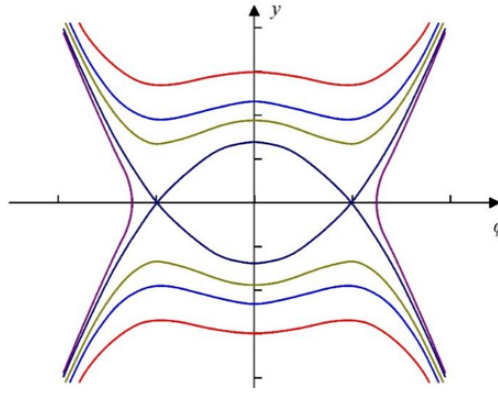


Figure 4: Phase diagram of the Hamiltonian system under condition 1

In order to find the dissimilar track connecting the two singular points $\left(\sqrt{-\frac{\alpha}{\beta}}, 0\right)$ and $\left(-\sqrt{-\frac{\alpha}{\beta}}, 0\right)$ in the diagram, the two points are substituted to obtain the equation:

$$\sqrt{\frac{-4h + 4c^2h + \frac{\alpha^2}{\beta}}{-1 + c^2}} = 0 \quad (50)$$

Solving for $h = -\frac{\alpha^2}{4(-1 + c^2)\beta}$, the desired isochronous track is:

$$y_2 = \sqrt{-\frac{(\alpha + \beta y_1^2)^2}{2(-1 + c^2)\beta}} \quad (51)$$

And:

$$y_2 = -\sqrt{-\frac{(\alpha + \beta y_1^2)^2}{2(-1 + c^2)\beta}} \quad (52)$$

Substituting (51) into the first equation of system (44) yields $\frac{dy_1}{d\xi} = \sqrt{-\frac{(\alpha + \beta y_1^2)^2}{2(-1 + c^2)\beta}}$. Therefore, the

parameter expression of (51) is

$$y_1 = f^{-1}\left(-\frac{\xi}{\sqrt{2}} + C\right), \quad \text{where } f(x) = \frac{\arctan\left(\frac{\beta}{\alpha}x\right)}{\sqrt{-\frac{\alpha(\alpha + \beta x^2)}{-1 + C^2}}}, \quad C \text{ are arbitrary constants.}$$

When $m > 0$ and $n < 0$, $\alpha\beta < 0$ and $c^2\alpha < \alpha$, the system (45) has three singularities $\left(\sqrt{-\frac{\alpha}{\beta}}, 0\right)$, $(0, 0)$ and $\left(-\sqrt{-\frac{\alpha}{\beta}}, 0\right)$. They are the center, saddle point and center, respectively. The phase diagram of the system (45) is shown in Figure 5.

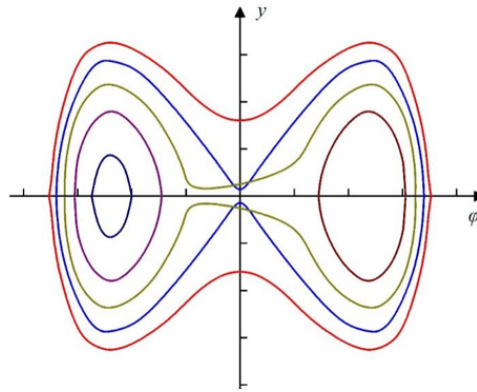


Figure 5: Phase diagram of the Hamiltonian system under condition 2

In order to find the graph connecting the singularity with the host rail, the point is substituted into the Hamilton function (46) to obtain the equation:

$$\frac{-4h + 4c^2h}{2(-1 + c^2)} = 0 \quad (53)$$

Solving for h , the desired homoclinic track is:

$$y_2 = \pm \sqrt{\frac{2\alpha y_1^2 + \beta y_1^4}{2 - 2c^2}}, \quad (y_1 < 0) \quad (54)$$

And:

$$y_2 = \pm \sqrt{\frac{2\alpha y_1^2 + \beta y_1^4}{2 - 2c^2}}, \quad (y_1 > 0) \quad (55)$$

Substituting (54) into the first equation of (46) yields $\frac{dy_1}{d\xi} = \pm \sqrt{\frac{2\alpha y_1^2 + \beta y_1^4}{2 - 2c^2}}$. So the parametric expression of (54) is $y_1(\xi) = f^{-1}\left(\pm \frac{\xi}{\sqrt{2}} + C\right)$, $y_1 < 0$. Where we have:

$$f(x) = \frac{(\ln x - \ln(2\alpha + \sqrt{2\alpha} \sqrt{2\alpha + \beta x^2})) x \sqrt{4\alpha + 2\beta x^2}}{2\sqrt{\alpha} \sqrt{\frac{2\alpha x^2 + \beta x^4}{1 - c^2}}} \quad (56)$$

C is an arbitrary constant.

When $n > 0$, $m \geq 0$ or $n = 0$, $m > 0$, $\beta > \beta c^2$, $\frac{\alpha}{1-c^2} \geq 0$ or $\beta = 0$, $\alpha > \alpha c^2$ system (45) has only one singularity $(0,0)$, which is the saddle point. The phase diagram of system (45) is shown in Figure 6(a). At this time, the system (45) does not have co-habitation track, heterodyne track and periodic track, so equation (44) does not have isolated waves and periodic waves.

When $n < 0$, $m \leq 0$ or $n = 0$, $m < 0$, $\beta < \beta c^2$, $\frac{\alpha}{1-c^2} \leq 0$ or $\beta = 0$, $\alpha < \alpha c^2$ system (45) has only one singularity $(0,0)$, which is the center. The phase diagram of the system (45) is shown in Figure 6(b). At this time, the system (45) does not have homo- and hetero-hosted tracks, so equation (44) does not have isolated and periodic waves. $H(y_1, y_2) = h$ is a family of periodic tracks of the system (45). To obtain its parametric expression, substitute (47) into (45) and solve for:

$$y_1(\xi) = f^{-1}\left(\frac{\xi}{\sqrt{2}} + C\right), \quad h > 0 \quad (57)$$

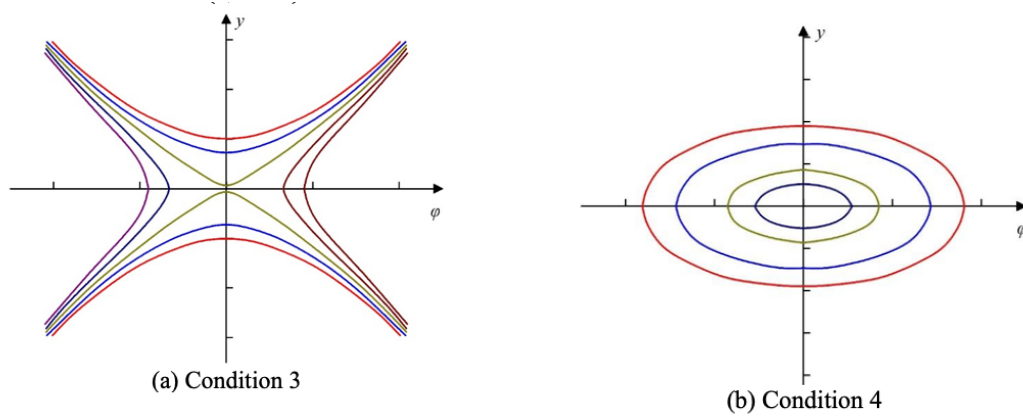


Figure 6: Phase diagram of the Hamiltonian system: (a) Condition 3; (b) Condition 4

5. Conclusion

This paper explores the traveling wave solutions and initial edge value problems of three multidimensional nonlinear mathematical physics equations, which are transformed into a planar dynamical system using certain treatments. For the first two equations, the branching phase diagrams corresponding to the planar system in different parameter cases are drawn using the branching theory of the dynamical system, and the exact expressions of their traveling wave solutions, including isolated wave solutions, periodic wave solutions, isolated sharp wave solutions and torsional waves, are calculated from the phase diagrams, so as to analyze the dynamical properties and intrinsic relationships of these solutions.

For nonlinear mathematical equations, the branching of the system solution and the corresponding solution structure can be analyzed by the integration process. However, there are not only perturbations to the parameters of the real system, but also perturbations to the structure of the system may exist. These structural perturbations, which may be linear, may also be nonlinear. They sometimes change the topology of the system phase diagram.

There is a rich dynamical behavior of multidimensional nonlinear mathematical physical equations, which is, in essence, consistent with the complexity of real systems. Therefore, to analyze each system in a fine way encounters more or less obstacles, and the research process in this paper highlights the unique role of the geometric theory of ordinary differential equations and branching theory in breaking through these obstacles.

Funding

Special Fund for Basic Scientific Research Expenses of Central Universities of Chang'an University (No.: 300102213212).

References

- [1] Biswas, A. (2020). Principal eigenvalues of a class of nonlinear integro-differential operators. *Journal of Differential Equations*, 268(9), 5257–5282. <https://doi.org/10.1016/j.jde.2020.01.008>
- [2] Hc, A., Cl, B., & Pw, A. (2020). A Nekhoroshev type theorem for the nonlinear wave equation. *Journal of Differential Equations*, 269(4), 3853–3889. <https://doi.org/10.1016/j.jde.2020.02.016>
- [3] Asadi, M., & Afshar, M. (2022). Fixed point theorems in the generalized rational type of C-class functions in b-metric spaces with application to integral equation. *3C Empresa: Investigación y Pensamiento Crítico*, 11(2), 64–74. <https://doi.org/10.17993/3cemp.2022.110249.64-74>
- [4] Suh, Y. J., Kim, G. J., & Woo, C. (2018). Pseudo anti-commuting Ricci tensor and Ricci soliton real hypersurfaces in complex hyperbolic two-plane Grassmannians. *Mathematische Nachrichten*. <https://doi.org/10.1002/mana.201700345>
- [5] Zhu, Y., Wei, S. C., Dong, Y. C., Liang, Y., & Wang, Y. J. (2017). The study of a novel ultrasonic A-scan signal processing method based on fractal theory. *Modern Physics Letters B*. <https://doi.org/10.1142/S0217984917400275>
- [6] Chen, D., Shi, S., Gu, X., & Shim, B. (2021). Weak signal frequency detection using chaos theory: A comprehensive analysis. *IEEE Transactions on Vehicular Technology*, 70(9), 7896–7905. <https://doi.org/10.1109/TVT.2021.3107839>
- [7] Wang, M., Li, X., & Zhang, J. (2017). The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*, 372(4), 417–423. <https://doi.org/10.1016/j.physleta.2007.07.006>
- [8] Seadawy, A., Ali, A., & Aljahdaly, N. (2020). The nonlinear integro-differential Ito dynamical equation via three modified mathematical methods and its analytical solutions. *Open Physics*, 18(1), 24–32. <https://doi.org/10.1515/phys-2020-0002>
- [9] Chai, Z., & Shi, B. (2020). Multiple-relaxation-time lattice Boltzmann method for the Navier-Stokes and non-linear convection-diffusion equations: Modeling, analysis, and elements. *Physical Review E*, 102(2), 023303. <https://doi.org/10.1103/PhysRevE.102.023303>
- [10] Liang, Z., & Li, B. (2017). Traveling wave solutions in an integro-differential competition model. *Discrete and Continuous Dynamical Systems - Series B*, 17(1), 417–428. <https://doi.org/10.3934/dcdsb.2017.17.417>
- [11] Al-Amr, M. O., & El-Ganaini, S. (2017). New exact traveling wave solutions of the (4+1)-dimensional Fokas equation. *Computers & Mathematics with Applications*, 74(6), 1274–1287. <https://doi.org/10.1016/j.camwa.2017.06.016>
- [12] Chan, H., & Wei, J. (2017). Traveling wave solutions for bistable fractional Allen-Cahn equations with a pyramidal front. *Journal of Differential Equations*, 262(9), 4567–4609. <https://doi.org/10.1016/j.jde.2017.01.006>
- [13] Yang, J., & Lou, B. (2017). Traveling wave solutions of competitive models with free boundaries. *Discrete and Continuous Dynamical Systems - Series B*, 19(3), 817–826. <https://doi.org/10.3934/dcdsb.2019039>
- [14] Kozlov, V. V. (2020). Quadratic conservation laws for equations of mathematical physics. *Russian Mathematical Surveys*, 75(3), 581–582. <https://doi.org/10.1070/RM9914>
- [15] Evikel, A. C. (2021). Soliton solutions of nonlinear fractional differential equations with its applications in mathematical physics. *Revista Mexicana de Física*, 67(3), 343–353. <https://doi.org/10.31349/RevMexFis.67.343>
- [16] Gaeta, G., & Spadaro, F. (2017). Random Lie-point symmetries of stochastic differential equations. *Journal of Mathematical Physics*. <https://doi.org/10.1063/1.4977836>
- [17] Gurses, M., & Pekcan, A. (2018). Nonlocal nonlinear Schrödinger equations and their soliton solutions. *Journal of Mathematical Physics*. <https://doi.org/10.1063/1.5023828>
- [18] Du, L., & Wu, Z. (2017). Solving the non-isentropic Navier-Stokes equations in odd space dimensions: The Green function method. *Journal of Mathematical Physics*, 58(10), 101506. <https://doi.org/10.1063/1.4998323>
- [19] Wang, C., & Su, J. (2022). The semilinear elliptic equations with double weighted critical exponents. *Journal of Mathematical Physics*, 63(3), 031504. <https://doi.org/10.1063/5.0079023>
- [20] Ramsey, S. D., & Baty, R. S. (2017). Symmetries of the gas dynamics equations using the differential form method. *Journal of Mathematical Physics*, 58(11), 111506. <https://doi.org/10.1063/1.5009253>
- [21] Luo, H., Tang, X., & Gao, Z. (2018). Ground state sign-changing solutions for fractional Kirchhoff equations in bounded domains. *Journal of Mathematical Physics*, 59(3), 031504. <https://doi.org/10.1063/1.5012748>