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Stochastic stability of a one-sided vibro-impact system excited by real noise

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Abstract

Vibro-impact is widespread in practical engineering, so it is of great practical significance to study it. However, the research on the stochastic stability and bifurcation of vibro-impact systems is still limited, especially the research on the moment stability. Based on the pth moment Lyapunov exponent, this paper studies the stochastic stability of the vibro-impact system driven by real noise. Firstly, the smooth stochastic dynamic system is obtained by using the non-smooth transformation, and the Ornstein-Uhlenbeck process is applied to characterize the real noise model. On this basis, the second-order approximate solution of the pth moment Lyapunov exponent is calculated by using the L.Arnold perturbation method, which is in good agreement with the simulation results of the Monte Carlo method. Finally, the influences of noise parameters, natural frequency, recovery coefficient and damping coefficient on the random stability of shock vibration are studied. Due to the existence of impact factors, the natural frequency has a direct and significant impact on the stochastic stability of the system.

Keywords: Vibro-impact system, Real noise, Moment Lyapunov exponent, Stochastic stability, Perturbation method

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1 Introduction

Vibro-impact is a commonly encountered phenomenon in real-world scenarios, relevant to various fields such as applied mathematics, applied physics, mechanical engineering, and others, significantly impacting our daily lives. The dynamics of vibro-impact systems become exceptionally complex due to the impact, revealing intriguing phenomena that distinguish them from smooth systems, such as torus bifurcation, grazing bifurcation, chatter, and sticking motion. Furthermore, random factors are commonly present in nature, such as earthquakes, strong winds, ocean waves, and atmospheric turbulence. Random factors can have destructive impacts on systems, such as the collapse of the Tacoma Narrows Bridge in the United States due to wind forces. Additionally, random factors can also exert positive influences on systems, for example, noise can induce phenomena like stochastic resonance, noise-induced transitions, phase transitions, and transport processes. Therefore, it is very necessary to study the influence of random factors on the dynamic characteristics of vibro-impact systems for practical engineering applications, which also has been investigated by many researchers [1-6].

In recent years, scholars in mathematics, mechanics, and engineering have dedicated significant efforts to studying the dynamic behavior of stochastic vibro-impact systems, yielding promising results[7-12]. As early as 1972, Nayak[13] conducted research on the stochastic dynamics of vibro-impact systems. Jing[14, 15] employed the Fokker-Planck-Kolmogorov equation method to derive the steady-state solution for vibro-impact systems excited by white noise. Feng[1, 16] examined the mean response of vibro-impact systems by constructing an averaged Poincaré map. Iourtchenko[17, 18] investigated the mean response of vibro-impact systems using energy averaging and energy balance methods. Huang[19] employed the stochastic averaging method to analyze the response problem in random vibro-impact systems. Namachchivaya[20] investigated the stochastic dynamics of a random vibro-impact system using an enhanced stochastic averaging method. Based on average energy dissipation, Gu[21] proposed a new method to investigate this issue and validated its feasibility with a numerical example. Liu[22] similarly utilized this method to study the stationary probability response of a vibro-impact system under real noise excitation, examining the effects of noise and restitution coefficient on system response. Meanwhile, Wu[23] employed perturbation methods to analyze the steady-state response of a vibro-impact system excited by Poisson white noise. Rong[24] utilized a multi-scale approach to study the nonlinear vibro-impact system responding to combined harmonic and random forces. Dimentberg[25] applied the path integral method to solve stochastic responses in vibro-impact systems with high collision losses. Using a novel path integral method, Wang[26, 27] examined the response and bifurcation in random vibro-impact systems. Recently, Chen[28-30] extended the radial basis function neural network method to non-smooth stochastic dynamical systems, and studied the stochastic response problem of vibro-impact systems. The random bifurcation problem in vibro-impact systems has garnered significant scholarly interest, resulting in numerous research achievements[31-34]. Xu[35] comprehensively summarize and comment on the research progress regarding the dynamic characteristics of vibro-impact systems.

The largest Lyapunov exponent is an important index to investigate the local stability and bifurcation behavior of stochastic dynamical systems. Feng[36] explored bifurcations in a vibro-impact system excited by white noise through calculation of the largest Lyapunov exponent. They also introduced Melnikov's method for a general vibro-impact oscillator[37]. Wang[38] assessed the almost sure stability of a vibro-impact system by computing the largest Lyapunov exponent. Using the stochastic averaging and Khasminskii methods, Xu[39] determines the largest Lyapunov exponent of a collision vibration system excited by additive and multiplicative Gaussian white noise, and discusses the influence of each parameter on the stochastic stability of the system. By calculating the largest Lyapunov exponent, Narayanan and Kumar[40-43] explores the stochastic bifurcation of vibro-impact systems with different random perturbations. According to the large deviation theory[44], it is noted that even if the random dynamical system is stable with probability 1, the *p*th moment of response could exhibit instability and grow exponentially. Therefore, investigating moment stability in a random dynamical system is crucial, which can be determined by the *p*th moment Lyapunov exponent. Compared to the largest Lyapunov exponent, the *p*th moment Lyapunov exponent offers a more comprehensive description of stochastic stability and stochastic bifurcation (D-bifurcation and P-bifurcation) behaviors in random dynamical systems, which is of great significance both in theory and engineering application. Despite its pivotal role in stochastic dynamical system research, calculating the *p*th moment Lyapunov exponent remains particularly challenging. In recent decades, researchers have made notable progress in investigating the *p*th moment Lyapunov exponents for smooth stochastic dynamical systems, yielding promising results[45-52]. However, for non-smooth random dynamical systems, the calculation of the *p*th moment Lyapunov exponents remains to be explored.

In the present paper, the *p*th moment Lyapunov exponent of a one-sided vibro-impact system excited by real noise is calculated, and the stochastic stability is investigated. Firstly, a dynamic model of a one-sided vibro-impact system is established, and it is simplified into an equivalent smooth system by dynamical non-smooth transformation. Then, applying the L.Arnold perturbation method, the second-order asymptotic analytic solution of the *p*th moment Lyapunov exponent is obtained. Finally, according to the stability index, the influence of parameters such as system noise intensity, natural frequency, damping coefficient and collision recovery coefficient on the stochastic dynamics of the system is discussed. On the basis of integrating and all the analytical results, the correlation between different parameters and the dynamics characteristics of the vibro-impact system is systematically explored, and a series of suggestions to improve the stochastic stability of the system are given.

2 Formula

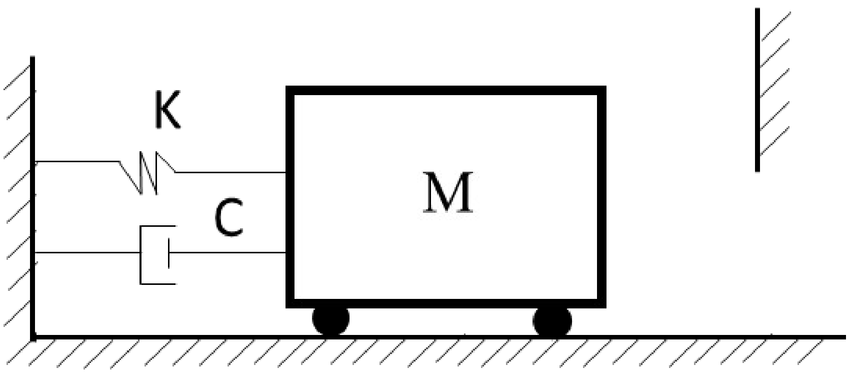


Fig. 1 Structure diagram of vibro-impact system

Without loss of generality, we investigate the vibro-impact system under random parameter excitation, illustrated in Fig. 1. The equations of motion are expressed as

$$\begin{aligned} &M\ddot{X}(\tau)+C\dot{X}(\tau)+K(1+\xi(\tau))X=0, \quad && X>0 \\ &\dot{X}_{+}=-r\dot{X}_{-}, && X=0 \end{aligned} \tag{1}$$

where X is the system displacement; K is the linear stiffness coefficient; $\dot{X}_+$ and $\dot{X}_-$ are the pre-impact and post-impact velocities, respectively; r is the coefficient of recovery; $\xi(\tau)$ is the real noise. The system is dimensionless by applying the following transformation

$$t=\omega\tau, \omega_0^2=K/(M\omega^2), c_1=c/M\omega \tag{2}$$

The system (1) is transformed:

$$\begin{cases} \ddot{x}+c_1\dot{x}+\omega_0^2(1+\xi(t))x=0, & x>0 \\ \dot{x}_+=-r\dot{x}_-, & x=0 \end{cases} \tag{3}$$

where $\dot{x}_+$ and are $\dot{x}_-$ the dimensionless velocities before and after the impact, respectively. The real noise $\xi(t)$ is an Ornstein-Uhlenbeck process, which is expressed as

$$d\xi=-\alpha_0\xi(t)dt+\sigma_0 dW(t) \tag{4}$$

The power spectral density of real noise is defined as follows

$$S(\omega)=\frac{\sigma_0^2}{\alpha_0^2+\omega^2} \tag{5}$$

Apply the following Zhuravlev transformation[53] to system (3):

$$\begin{cases} x=|y| \\ \dot{x}=\dot{y}\,\mathrm{sgn}(y) \\ \ddot{x}=\ddot{y}\,\mathrm{sgn}(y) \end{cases} \tag{6}$$

where

$$\mathrm{sgn}(y)=\begin{cases} 1, & y>0 \\ 0, & y=0 \\ -1, & y<0 \end{cases}$$

The system (3) is converted to

$$\ddot{y}+\omega_0^2 y+\omega_0^2\xi(t)y+c_1\dot{y}+(1-r)\dot{y}|\dot{y}|\delta(y)=0 \tag{7}$$

where y is the transformed dimensionless displacement variable and $\delta(y)$ is the Dirac function. In order to investigate the stochastic stability of the vibro-impact system analytically, we assume the random term is a small quantity of parameter ε ($0<\varepsilon<<1$); the impact energy loss and damping term are both small quantity of parameter ε^2. Then, the system (7) becomes

$$\ddot{y}+\omega_0^2 y+\varepsilon\omega_0^2\xi(t)y+\varepsilon^2 c_1\dot{y}+\varepsilon^2(1-r)\dot{y}|\dot{y}|\delta(y)=0 \tag{8}$$

3 Moment Lyapunov exponent

The *p*th moment Lyapunov exponent is an important index for studying the stochastic stability and bifurcation of dynamical systems, which is defined as

$$\Lambda(p)=\lim_{t\to\infty}\frac{1}{t}\log \mathbf{E}\left[\left\|\mathbf{x}(t;\mathbf{x}_0)\right\|^p\right] \tag{9}$$

where $\mathbf{x}(t;\mathbf{x}_0)$ is the solution process of a stochastic dynamical system, and $\mathbf{x}_0$ is the initial value. Based on the definition of *p*th moment Lyapunov exponent, it can be concluded that if $\Lambda(p)<0$, then $\mathbf{E}\left[\left\|\mathbf{x}(t;\mathbf{x}_0)\right\|^p\right]\to 0$ as $t\to\infty$, indicating that the dynamical system is *p*th moment stable. The *p*th moment Lyapunov exponent $\Lambda(p)$ has a unique nonzero solution δ_p, i.e., $\Lambda(\delta_p)=0$, where δ_p is referred to as the moment stability index[54]. Furthermore, based on the *p*th moment Lyapunov exponent $\Lambda(p)$, the largest Lyapunov exponent λ of the system can be obtained[55].

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\log\left\|\mathbf{x}(t;\mathbf{x}_0)\right\|=\frac{\mathrm{d}}{\mathrm{d}p}\Lambda(p)\Big|_{p=0} \tag{10}$$

3.1 Results based on perturbation method

Let $y=y_1$, $\dot{y}=\omega_0 y_2$, convert the system (8) into the following form

$$\begin{aligned} dy_1&=\omega_0 y_2 dt\\ dy_2&=(-\omega_0 y_1+\varepsilon^2\left(-c_1+(r-1)\omega_0\left|y_2\right|\delta(y_1)\right)y_2-\varepsilon\omega_0\xi(t)y_1)dt\\ d\xi&=-\alpha_0\xi(t)dt+\sigma_0 dW(t) \end{aligned} \tag{11}$$

Let $y_1=e^{\rho}\cos\varphi$, $y_2=-e^{\rho}\sin\varphi$, $\varphi\in[0,2\pi]$, and the polar coordinate form of system (11) is given as follows

$$\begin{aligned} d\rho&=\left(\varepsilon^2 q_2(\varphi)+\varepsilon q_1(\varphi)\xi(t)\right)dt\\ d\varphi&=\left(\omega_0+\varepsilon^2 h_2(\varphi)+\varepsilon h_1(\varphi)\xi(t)\right)dt\\ d\xi&=-\alpha_0\xi(t)dt+\sigma_0 dW(t) \end{aligned} \tag{12}$$

where

$$q_1(\varphi)=\frac{1}{2}\omega_0\sin 2\varphi, \qquad h_1=\frac{1}{2}\omega_0(1+\cos 2\varphi)$$
$$q_2(\varphi)=\frac{1}{2}\left(-c_1+(r-1)\omega_0\left|\sin\varphi\right|\delta(\cos\varphi)\right)(1-\cos 2\varphi)$$
$$h_2(\varphi)=\frac{1}{2}\left(-c_1+(r-1)\omega_0\left|\sin\varphi\right|\delta(\cos\varphi)\right)\sin 2\varphi$$

According to Eq. (12), one easily finds that the random process $\varphi(t)$ is independent of the variable ρ. Thus, the random process $\varphi(t)$ alone forms a diffusive Markov process with the following generator

$$L_\varepsilon(p)=L_0(p)+\varepsilon L_1(p)+\varepsilon^2 L_2(p) \tag{13}$$

where

$$L_0(p)=-\alpha_0\xi\frac{\partial}{\partial\xi}+\frac{1}{2}\sigma_0^2\xi\frac{\partial^2}{\partial\xi^2}+\omega_0\frac{\partial}{\partial\varphi}$$
$$L_1(p)=h_1(\varphi)\xi\frac{\partial}{\partial\varphi}+pq_1(\varphi)\xi$$
$$L_2(p)=h_2(\varphi)\frac{\partial}{\partial\varphi}+pq_2(\varphi)$$

From the literatures [46, 54, 55] , the pth moment Lyapunov exponent $\Lambda(p)$ is the largest eigenvalue of the operator $L_\varepsilon(p)$. i.e.,

$$L_\varepsilon(p)T(p)=\Lambda(p)T(p) \tag{14}$$

Expand the Lyapunov exponent $\Lambda(p)$ and the eigenfunction $T(p)$ hierarchically:

$$\begin{aligned}\Lambda(p)&=\Lambda_0(p)+\varepsilon\Lambda_1(p)+\varepsilon^2\Lambda_2(p)+\cdots+\varepsilon^n\Lambda_n(p)+\cdots\\ T(p)&=T_0(p)+\varepsilon T_1(p)+\varepsilon^2 T_2(p)+\cdots+\varepsilon^n T_n(p)+\cdots\end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (14) yields

$$\varepsilon^0:\left(L_0(p)-\Lambda_0(p)\right)T_0(p)=0 \tag{16}$$

$$\varepsilon^1:\left(L_0(p)-\Lambda_0(p)\right)T_1(p)=\left(\Lambda_1(p)-L_1(p)\right)T_0(p) \tag{17}$$

$$\begin{aligned}&\varepsilon^2:\left(L_0(p)-\Lambda_0(p)\right)T_2(p)\\ &=\left(\Lambda_1(p)-L_1(p)\right)T_1(p)+\left(\Lambda_2(p)-L_2(p)\right)T_0(p)\end{aligned} \tag{18}$$

3.1.1 Zeroth-order perturbation

By definition (9), it is known that for any p

$$\Lambda_0(p) \equiv 0 \tag{19}$$

Thus, Eq. (16) is simplified to

$$-\alpha_0 \xi \frac{\partial T(p)}{\partial \xi} + \frac{1}{2}\sigma_0^2 \xi \frac{\partial^2 T(p)}{\partial \xi^2} + \omega_0 \frac{\partial T(p)}{\partial \varphi} = 0 \tag{20}$$

Applying the method of separated variables, let $T_0(p) = \Phi(\varphi)\Psi(\xi)$, and satisfy

$$\frac{\dot{\Phi}}{\Phi} = C, \qquad -\alpha\xi \frac{\dddot{\Psi}}{\Psi} + \frac{1}{2}\sigma_0^2 \frac{\Psi}{\Psi} = -\omega_0 C \tag{21}$$

From the periodic boundary condition $\Phi(\varphi + 2\pi) = \Phi(\varphi)$, it is easy to see that $C = 0$, and $\Phi(\varphi)$ is a constant. Thus

$$-\alpha\xi \frac{\dddot{\Psi}}{\Psi} + \frac{1}{2}\sigma_0^2 \frac{\Psi}{\Psi} = 0 \tag{22}$$

We can obtain:

$$\Psi(\xi) = C_1 + C_2 erf\left(i \frac{\sqrt{\alpha_0}}{\sigma_0} \xi \right) \tag{23}$$

where $erf(\bullet)$ is the error function; j stands for the imaginary unit. Because $\Xi(\xi)$ is bounded as $\xi \to \infty$, so we get that $C_2 = 0$ i.e., $\Xi(\xi)$ is a constant. Therefore, $T_0(p)$ is a constant, i.e., $T_0(p) = C$.

The adjoint equation of Eq. (20) is

$$L_0^*(p) T_0^*(p) = 0 \tag{24}$$

By applying a similar method to solve Eq. (24), one easy to obtain that

$$T_0^*(p) = \frac{P_s(\xi)}{2\pi C} \tag{25}$$

where $P_s(\xi)$ is the steady-state probability density function of the stochastic process $\xi(t)$.

3.1.2 First-order perturbation

Based on the results of section 3.1.1, Eq. (17) can be simplified as

$$L_0(p)T_1(p)=(\Lambda_1(p)-L_1(p))T_0(p)=(\Lambda_1(p)-pq_1(\varphi)\xi)T_0(p) \tag{26}$$

Its solvability condition is

$$\begin{aligned}&\langle(\Lambda_1(p)-L_1(p))T_0(p),T_0^*(p)\rangle\\&=\int_0^{2\pi}\int_{\infty}^{+\infty}(\Lambda_1(p)-L_1(p))T_0(p)T_0^*(p)d\xi d\varphi=0\end{aligned} \tag{27}$$

It is obtained from Eq. (27)

$$\Lambda_1(p)=\frac{1}{2\pi}\int_0^{2\pi}pq_1(\varphi)\int_{\infty}^{+\infty}\xi P_s(\xi)d\xi d\varphi=0 \tag{28}$$

And so we have

$$L_0(p)T_1(p)=-pq_1(\varphi)\xi T_0(p) \tag{29}$$

i.e.,

$$\left(-\alpha_0\xi\frac{\partial}{\partial\xi}+\frac{1}{2}\sigma_0^2\xi\frac{\partial^2}{\partial\xi^2}+\omega_0\frac{\partial}{\partial\varphi}\right)T_1(p)=-pC\xi q_1(\varphi) \tag{30}$$

The solution[46, 47, 56] of Eq. (30) is

$$T_1(p)=T_1(\varphi,\xi,p)=Cp\int_0^{+\infty}q_1(\varphi+\omega_0\tau)K(\xi,\tau)d\tau \tag{31}$$

where $K(\xi,\tau)=\int_{\infty}^{+\infty}\xi_\tau P(\xi_\tau,\tau;\xi,0)d\xi_\tau$, $P(\xi_\tau,\tau;\xi,0)$ is the transient density.

3.1.3 Second-order perturbation

Using the existing results, Eq. (18) is simplified to

$$L_0(p)T_2(p)=C(\Lambda_2(p)-pq_2(\varphi))-L_1(p)T_1(p) \tag{32}$$

Its solvability condition is

$$\begin{aligned}&\langle C(\Lambda_2(p)-pq_2(\varphi))-L_1(p)T_1(p),T_0^*(p)\rangle\\&=\int_0^{2\pi}\int_{\infty}^{+\infty}(\Lambda_2(p)-pq_2(\varphi))\frac{P_s(\xi)}{2\pi}-L_1(p)T_1(p)\frac{P_s(\xi)}{2\pi C}d\xi d\varphi\end{aligned} \tag{33}$$

So we can get

$$\begin{aligned}\Lambda_2(p)&=\frac{1}{2\pi}\int_0^{2\pi}pq_2(\varphi)d\varphi+\int_0^{2\pi}\int_{\infty}^{+\infty}L_1(p)T_1(p)\frac{P_s(\xi)}{2\pi C}d\xi d\varphi\\&=\frac{p}{16}\left((p+2)\omega_0^2S(2\omega_0)+8\left(-c_1+\frac{2\omega_0(r-1)}{\pi}\right)\right)\end{aligned} \tag{34}$$

where $S(\omega_0)$ is the power spectral density of the stochastic process $\xi(t)$ with the following relation

$$S(\omega_0)=2\int_0^{+\infty}R(\tau)\cos(\omega_0\tau)d\tau,\quad R(\tau)=\int_{\infty}^{+\infty}\xi P_s(\xi)K(\xi,\tau)d\xi \tag{35}$$

Since $\Lambda_0(p)=0$, $\Lambda_1(p)=0$, and ε is a small parameter, the *p*th moment Lyapunov exponent is approximately given as

$$\begin{aligned}\Lambda(p)&\simeq\varepsilon^2\Lambda_2(p)\\&=\frac{\varepsilon^2}{16}p\left((p+2)\omega_0^2S(2\omega_0)+8\left(-c_1+\frac{2\omega_0(r-1)}{\pi}\right)\right)\end{aligned} \tag{36}$$

The system stability index is

$$\delta_p=\frac{1}{\omega_0^2S(2\omega_0)}\left(\frac{16\omega_0}{\pi}(1-r)+8c_1-2\omega_0^2S(2\omega_0)\right) \tag{37}$$

According to definition (10), the largest Lyapunov exponent is obtained

$$\lambda=\frac{\varepsilon^2}{8}\left(\omega_0^2S(2\omega_0)+4\left(-c_1+\frac{2\omega_0(r-1)}{\pi}\right)\right) \tag{38}$$

4. Results analysis

To validate the reliability of the approximate analytical solutions of the moment Lyapunov exponent derived via L. Arnold's perturbation method, numerical calculations of the moment Lyapunov exponent for the random vibro-impact system (3) are performed using the Monte Carlo method [57]. Fig. 2 presents the analytical and numerical results of the *p*th moment Lyapunov exponent across varying noise parameters. As depicted in Fig. 2, the negligible difference between the analytical and numerical results confirms the accuracy of the perturbation-based approximate analytical solution of the moment Lyapunov exponent. Building on these analytical results of the moment Lyapunov exponent $\Lambda(p)$ and the largest Lyapunov exponent λ, a detailed analysis of the stochastic stability of vibro-impact system (3) is subsequently carried out. Equations (36-38) reveal that the stochastic stability of system (3) is governed by the power spectral density $S(\omega_0)$, coefficient of restitution r, damping coefficient c_1, and natural frequency ω_0

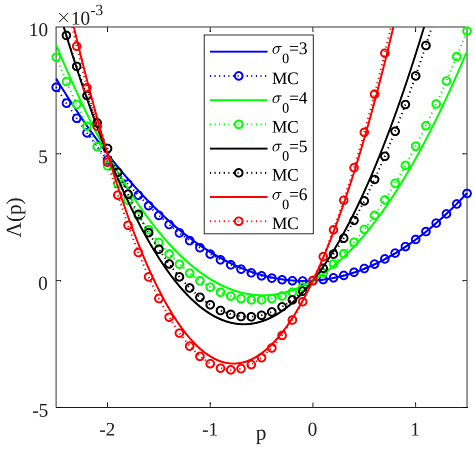


Fig. 2 Analytical and numerical solutions of moment Lyapunov exponents for the case
$\varepsilon=0.1, \omega_0=\pi, r=0.8, c_1=0.1; \alpha_0=1.$

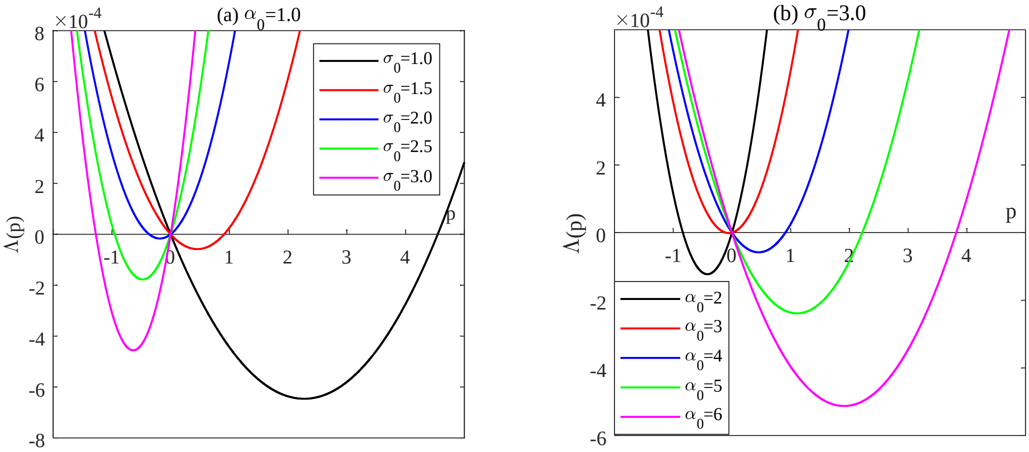


Fig. 3 Effect of noise parameters on the *p*th moment Lyapunov exponent for the case
$\varepsilon=0.1, \omega_0=1, r=0.9, c_1=0.1.$

Fig. 3 and Fig. 4 respectively present the influence of noise parameters on the *p*th moment Lyapunov exponent and the largest Lyapunov exponent. The intersection of $\Lambda(p)$ with the x-axis (i.e., $\Lambda(p)=0$) is the system's stability indicator that characterizes the stability of the system, and the larger the *p*-value corresponding to the location of its intersection, the more stable the system will be over a larger range of parameters. Fig. 3(a) and Fig. 4(a) show that as the diffusion parameter σ_0 increase, the random stability of the system is gradually weakening, and even the system may become unstable. On the contrary, the larger the drift parameter α_0 is, the more stable the system is, as shown in Fig. 3(b) and Fig. 4(b).

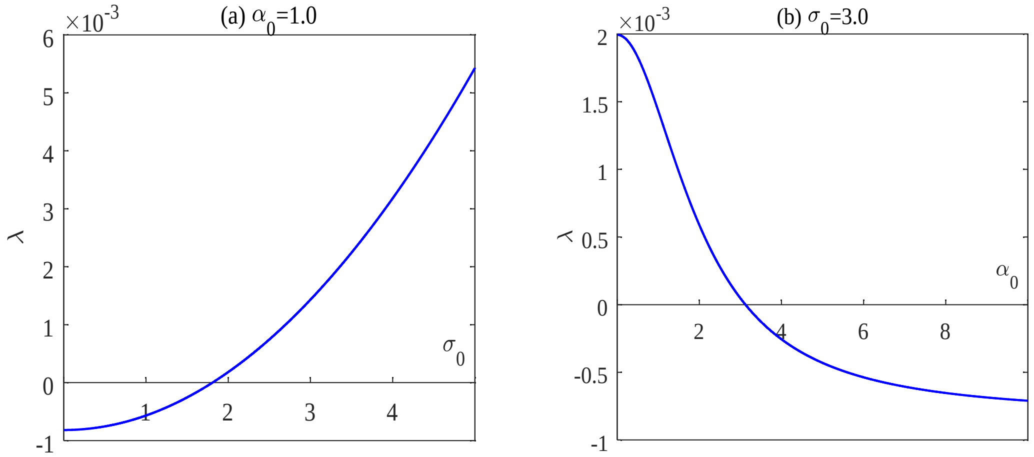


Fig. 4 Effect of noise parameters on the largest Lyapunov exponent for the case

$\varepsilon = 0.1, \omega_0 = 1, r = 0.9, c_1 = 0.1.$

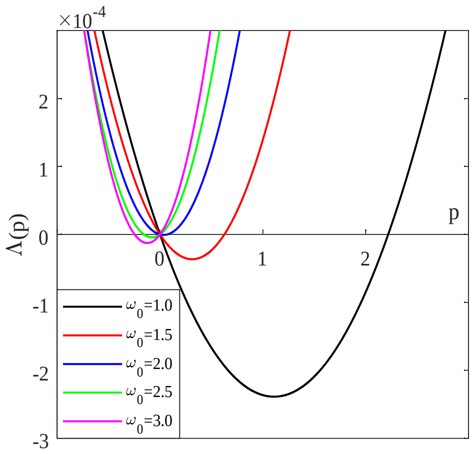
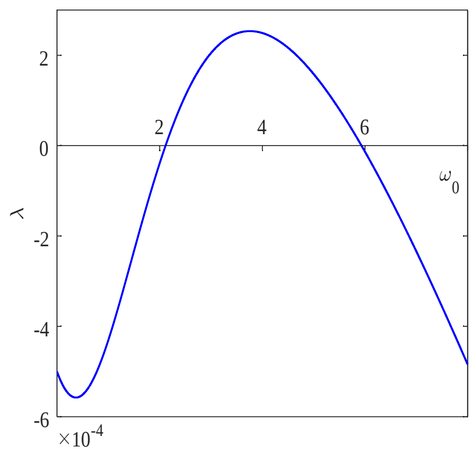


Fig. 5 Effect of natural frequency ω_0 on the *p*th moment Lyapunov exponent for the case

$\varepsilon = 0.1, r = 0.9, c_1 = 0.1; \alpha_0 = 5, \sigma_0 = 3.$



ω_0

Fig. 6 Effect of natural frequency on the largest Lyapunov exponent for the case

$\varepsilon = 0.1, r = 0.9, c_1 = 0.1; \alpha_0 = 5, \sigma_0 = 3.$

The effects of the natural frequency ω_0 on the *p*th moment Lyapunov exponent and largest Lyapunov exponent are depicted in Fig. 5 and Fig. 6, respectively. In Fig. 5, it can be found that the system stability index becomes progressively smaller with the increase of the natural frequency, and the corresponding stabilization interval is also shrinking. From Fig. 6, it can be observed that the largest Lyapunov exponent is nonlinearly related to the natural frequency. According to Eqs. (36-38), it can be inferred that there is a nonlinear nonmonotonic relationship between the natural frequency and the stochastic stability of the system.

Fig. 7 presents the relationship curve between the *p*th moment Lyapunov exponent and the recovery coefficient *r*. From the Fig. 7, it can be seen that as the parameter *r* increases, the curve of $\Lambda(p)$ is gradually shifting upward and the corresponding stability interval is shrinking, which indicates that the system is more likely to exhibit unstable behavior at higher recovery coefficients. According to the largest Lyapunov exponent calculation result (38), the critical value of *r* is expressed as $1+\pi c_1/2\omega_0 - \pi\omega_0\sigma_0^2/8\left(\alpha_0^2+4\omega_0^2\right)$, when $r < 1+\pi c_1/2\omega_0 - \pi\omega_0\sigma_0^2/8\left(\alpha_0^2+4\omega_0^2\right)$, the system is probability 1 stable; when $r > 1+\pi c_1/2\omega_0 - \pi\omega_0\sigma_0^2/8\left(\alpha_0^2+4\omega_0^2\right)$, the system is probability 1 unstable. In summary, it can be seen that reducing the system recovery coefficient can enhance the system stability.

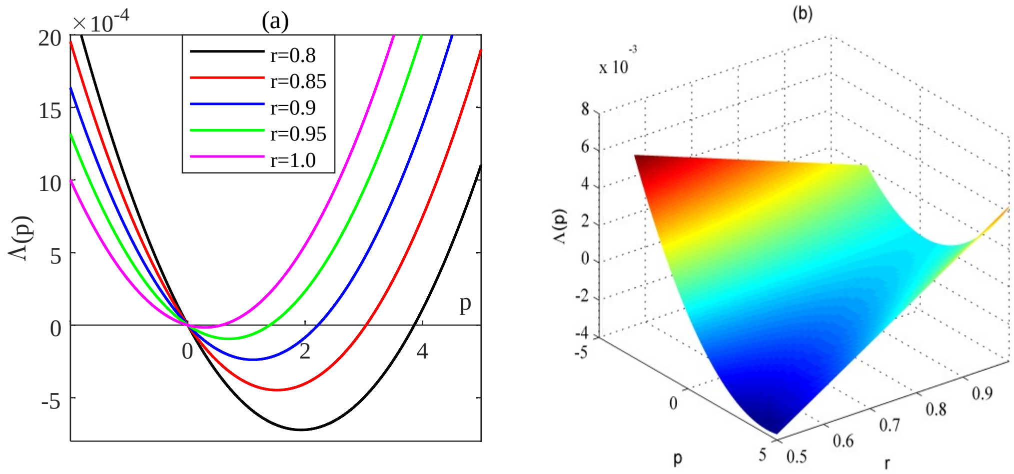


Fig. 7 Effect of recovery coefficient *r* on the *p*th moment Lyapunov exponent for the case $\varepsilon=0.1, \omega_0=0.1, c_1=0.1; \alpha_0=5, \sigma_0=3.$

Fig. 8 plots the influence of the damping coefficient c_1 on the *p*th moment Lyapunov exponent. It can be seen from the Fig. 8 that with the increase of damping coefficient, the unstable region of the system becomes more extensive. From equation (38), the critical value is expressed as $\sigma_0^2\omega_0^2/4\left(\alpha_0^2+4\omega_0^2\right)+2\omega_0^2(r-1)/\pi$, i.e., when $c_1>\sigma_0^2\omega_0^2/4\left(\alpha_0^2+4\omega_0^2\right)+2\omega_0^2(r-1)/\pi$ the system is probability 1 stable; on the contrary, when $c_1<\sigma_0^2\omega_0^2/4\left(\alpha_0^2+4\omega_0^2\right)+2\omega_0^2(r-1)/\pi$ the stochastic dynamical system is probability 1 unstable. In summary, increasing the damping coefficient of the system can enhance the system stability.

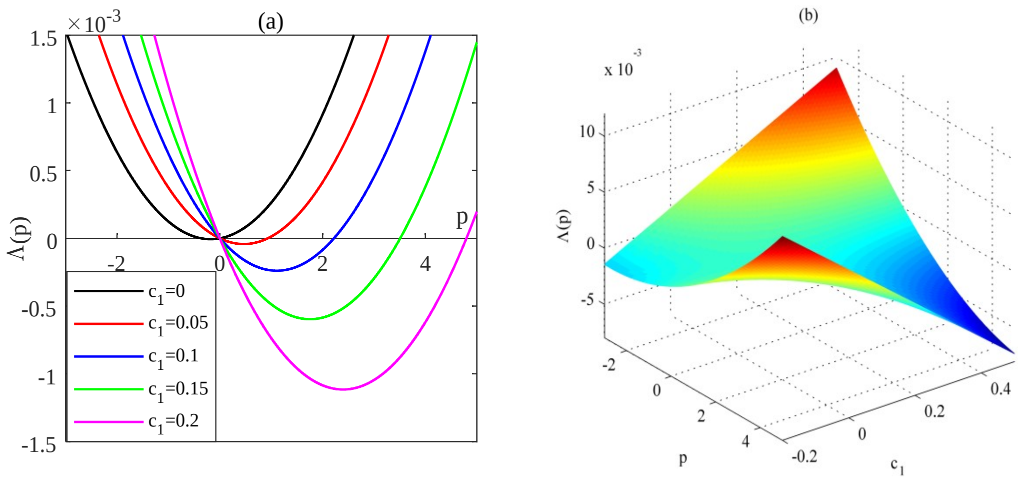


Fig. 8 Effect of damping coefficient c_1 on *p*th moment Lyapunov exponent for the case $\varepsilon=0.1, \omega_0=0.1, r=0.9; \alpha_0=5, \sigma_0=3.$

6. Conclusions

This paper delves into the stochastic stability of a vibro-impact system subjected to real noise excitation by calculating the *p*th moment Lyapunov exponent. The vibro-impact system is converted into a smooth stochastic dynamic system through the application of non-smooth transformation. The Ornstein-Uhlenbeck process is utilized to characterize the real noise model. Building on this foundation, the second-order approximate solution of the *p*th moment Lyapunov exponent is derived using the L.Arnold perturbation method. This solution exhibits a high degree of consistency with the simulation outcomes obtained from the Monte Carlo method, thereby validating the accuracy and effectiveness of the proposed approach.

This research further explores the impacts of various parameters on the stochastic stability of vibro-impact systems. The parameters under consideration include noise parameters, natural frequency, recovery coefficient, and damping coefficient. The findings indicate that the presence of impact factors means the natural frequency exerts a direct and substantial influence on the system's stochastic stability.

The achievements of this study provide a robust theoretical foundation for comprehending and forecasting the stochastic stability of impact vibration systems. They hold significant reference value for addressing practical engineering problems and establish a basis for future in-depth research into the stochastic dynamic behavior of vibro-impact systems.

Looking ahead, avenues for further research could involve examining the effects of more intricate factors on system stability or investigating the stability characteristics of different types of vibro-impact systems across diverse noise environments, among other possibilities.

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