

Global stability of linear systems with nonlinear perturbations: quadratic thresholds and the real stability radius gap

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Abstract

Let A be Hurwitz and f locally Lipschitz with $f(0) = 0$. Which explicit conditions on f make the origin of $\dot{x} = Ax + f(x)$ globally asymptotically stable? Using quadratic Lyapunov functions $V = \langle Px, x \rangle$, we show that V is a strict certificate exactly when f lies in a pointwise half-space, and that the largest certified sector $\|f(x)\| \leq \gamma \|x\|$ is $\gamma < r_{\mathbb{C}}(A)$, the complex stability radius. Our new results concern perturbations not confined to a ball: a direction $h(x)Mx$ is certified for every scalar h if and only if M is skew for some P with $A^{\top}P + PA < 0$, and a gain–dissipation trade-off allows unbounded gain where local dissipation compensates it. We then compare $r_{\mathbb{C}}$ with the real stability radius $r_{\mathbb{R}}$: they coincide for normal matrices, or when the supremum defining $r_{\mathbb{C}}$ is attained at $\omega = 0$, and otherwise, for an explicit 3×3 example, nonlinear perturbations destabilize strictly below $r_{\mathbb{R}}$, bracketing the exact threshold in $[0.1671, 0.2001]$. We also prove that r_{NL} equals the radius r_{TV} for time-varying perturbations when $n \geq 3$, which yields an exact criterion for when the quadratic threshold is the global one, and that for A_3 the upper bound can be taken as 0.2000007. We also give averaged-gain and switching theorems with exact rate functions, which recover the sharp duty cycle in the burst examples, and discuss applications.

Keywords: Global asymptotic stability, Lyapunov function, Riccati inequality, stability radius, S-lemma, input-to-state stability

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1 Introduction

Let $A \in \mathbb{R}^{n \times n}$ be Hurwitz and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ locally Lipschitz with $f(0) = 0$. We study

$$\dot{x} = Ax + f(x). \quad (1)$$

If $f(x) = o(\|x\|)$, the origin is locally exponentially stable. The question is which *explicit* conditions on f give *global* asymptotic stability. Some condition is indispensable: $\dot{x} = -x + x^2$ has finite escape time for $x_0 > 1$ (with $u = 1/x$, $\dot{u} = u - 1$, so $x(t) = 1/(1 + (1/x_0 - 1)e^t)$).

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The textbook answer [1, 2] solves $A^\top P_0 + P_0 A = -I$ and requires $\|f(x)\| \leq \gamma \|x\|$ with $\gamma < 1/(2\lambda_{\max}(P_0))$. It has two weaknesses: P_0 is fixed, and f is confined to a ball, hence to linear growth. We remove both.

Write $Q_P := -(A^\top P + PA)$ and $\mathcal{P}(A) := \{P = P^\top > 0 : Q_P > 0\}$ (a nonempty convex cone).

Contributions

1. *Exact quadratic certificates* (Theorem 3, Proposition 4). $V = \langle Px, x \rangle$ is a strict certificate iff f satisfies a half-space condition; the largest ball $\|f\| \leq \gamma \|x\|$ certified by one quadratic form is $\gamma < r_{\mathbb{C}}(A)$.
2. *Certified directions of unbounded size* (Theorem 6, Proposition 7). $h(x)Mx$ is certified for every h iff $PM + M^\top P = 0$ for some $P \in \mathcal{P}(A)$; in the plane the Euclidean rotation $h(x)\Omega x$ qualifies iff $A + A^\top < 0$.
3. *Gain–dissipation trade-off* (Theorem 10) with time dependence and ISS.
4. *Optimality* (Theorem 12, Proposition 14, Theorem 19). The exact global threshold for sector perturbations is $r_{\mathbb{C}}(A) = r_{\mathbb{R}}(A)$ for normal A or when the maximizing frequency is 0; otherwise nonlinear perturbations destabilize strictly below $r_{\mathbb{R}}(A)$ in an explicit example.
5. *Extensions* (Section 7): averaged gain and switching between certificates, with exact rate functions (sharp in the burst examples), and *applications* (Section 9).

Relation to the literature

Item 1 is essentially classical: the identification of the complex stability radius with a Riccati inequality is due to Hinrichsen and Pritchard [8, 9, 11], and the equivalence between quadratic stability under norm-bounded time-varying uncertainty and an H_∞ bound to Khargonekar, Petersen and Zhou [12, 13]. We include them with short proofs, since they define what a quadratic V can deliver and are the starting point for the rest. Stability radii for nonlinear and time-varying perturbations are studied in [11, 14–16], differences between real and complex radii in [10], and the failure of Aizerman- and Kalman-type conjectures in [17–19]. Computation of the real stability radius goes back to [7] and is the subject of recent work on large-scale and Frobenius-norm variants [20, 21]; the S-lemma behind our Proposition 4 has recently been extended to non-Euclidean norms [22]. The framework is close to passivity and port-Hamiltonian theory [6]. We did not locate Items 2–3 or the explicit examples of Item 4 in the literature, which does not exclude that related statements exist.

Notation

$\|\cdot\|$ is the Euclidean (spectral) norm; $\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2 Preliminaries

For Hurwitz A , the *complex stability radius* is

$$r_{\mathbb{C}}(A) = \left(\sup_{\omega \in \mathbb{R}} \|(i\omega I - A)^{-1}\| \right)^{-1},$$

and the *real stability radius* $r_{\mathbb{R}}(A) \geq r_{\mathbb{C}}(A)$ is the smallest norm of a real Δ with $A + \Delta$ having an eigenvalue on $i\mathbb{R}$ [4].

Lemma 1 (Bounded real lemma, strict form [3, Ch. 21]). *There is $P = P^\top > 0$ with $A^\top P + PA + P^2 + \gamma^2 I < 0$ iff $\gamma < r_{\mathbb{C}}(A)$.*

Lemma 2 (S-lemma [5]). *Let F, G be quadratic forms on $\mathbb{R}^m$ with $G(z_0) > 0$ for some z_0 . Then $F(z) \leq 0$ whenever $G(z) \geq 0$ iff there is $\tau \geq 0$ with $F + \tau G \leq 0$ on $\mathbb{R}^m$.*

3 Quadratic certificates: the exact condition

Theorem 3. Let $P \in \mathcal{P}(A)$ and $V(x) = \langle Px, x \rangle$. Then $\dot{V} \leq -\varepsilon \|x\|^2$ along (1) for some $\varepsilon > 0$ iff

$$2\langle Px, f(x) \rangle \leq \langle Q_P x, x \rangle - \varepsilon \|x\|^2 \quad \forall x \in \mathbb{R}^n. \quad (2)$$

In that case solutions exist for all $t \geq 0$ and $\|x(t)\| \leq \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)} e^{-\mu t} \|x(0)\|$, $\mu = \varepsilon/(2\lambda_{\max}(P))$.

Proof. $\dot{V} = -\langle Q_P x, x \rangle + 2\langle Px, f(x) \rangle$ gives the equivalence. V is radially unbounded and nonincreasing, so there is no finite escape; $\dot{V} \leq -(\varepsilon/\lambda_{\max}(P))V$ and comparison give the bound.

Condition (2) confines $f(x)$ to a *half-space*: only the component of f along Px matters; the orthogonal component is free. This is why ball conditions are conservative.

Proposition 4 (Exactness of $r_{\mathbb{C}}(A)$). For $\gamma \geq 0$ the following are equivalent:

- (i) there are $P > 0$, $\varepsilon > 0$ with $A^\top P + PA + P^2 + \gamma^2 I \leq -\varepsilon I$;
- (ii) there is $P \in \mathcal{P}(A)$ such that every f with $\|f(x)\| \leq \gamma \|x\|$ satisfies (2) with a common $\varepsilon' > 0$;
- (iii) $\gamma < r_{\mathbb{C}}(A)$.

Proof. (i) $\Leftrightarrow$ (iii) is Lemma 1. (i) $\Rightarrow$ (ii): if $\|w\| \leq \gamma \|x\|$ then $2\langle Px, w \rangle \leq \|Px\|^2 + \|w\|^2 \leq \langle P^2 x, x \rangle + \gamma^2 \|x\|^2 \leq \langle Q_P x, x \rangle - \varepsilon \|x\|^2$; taking $w = 0$ gives $Q_P > 0$.

(ii) $\Rightarrow$ (i): let $z = (x, w)$, $F(z) = 2\langle Px, w \rangle - \langle Q_P x, x \rangle + \varepsilon' \|x\|^2$, $G(z) = \gamma^2 \|x\|^2 - \|w\|^2$. For $x \neq 0$, $\|w\| \leq \gamma \|x\|$, the linear map $f(y) = \frac{\langle y, x \rangle}{\|x\|^2} w$ is admissible with $f(x) = w$, so $F \leq 0$ on $\{G \geq 0\}$. As $G(x, 0) > 0$ for $x \neq 0$, Lemma 2 gives $\tau \geq 0$ with $\begin{pmatrix} -Q_P + (\varepsilon' + \tau\gamma^2)I & P \\ P & -\tau I \end{pmatrix} \leq 0$. If $\tau = 0$ the (2, 2) block vanishes and forces $P = 0$; so $\tau > 0$, and the Schur complement yields $-Q_P + (\varepsilon' + \tau\gamma^2)I + P^2/\tau \leq 0$. Dividing by τ and setting $\tilde{P} = P/\tau$ (note $Q_{\tilde{P}} = Q_P/\tau$) gives $A^\top \tilde{P} + \tilde{P}A + \tilde{P}^2 + \gamma^2 I \leq -(\varepsilon'/\tau)I$.

Thus $r_{\mathbb{C}}(A)$ is the exact threshold of sector conditions certified by one quadratic form. This is known [8, 12]; the proof above is a short self-contained derivation.

Example 5 (Improvement over the classical bound). Let $A = \begin{pmatrix} -1 & 10 \\ 0 & -1 \end{pmatrix}$ and $P_0 = \begin{pmatrix} 0.5 & 2.5 \\ 2.5 & 25.5 \end{pmatrix}$ ($A^\top P_0 + P_0 A = -I$, $\lambda_{\max}(P_0) = 25.75$). The classical bound is 0.0194; $r_{\mathbb{C}}(A) = 0.09902$, a factor 5.1 better.

4 Certified directions of unbounded size

Balls force linear growth. We now ask which *directions* Mx can be multiplied by an arbitrary scalar function without losing the certificate.

Theorem 6 (Certified directions). Let $P \in \mathcal{P}(A)$ and $M \in \mathbb{R}^{n \times n}$. The following are equivalent:

- (a) for every locally Lipschitz $h: \mathbb{R}^n \rightarrow \mathbb{R}$, $V = \langle Px, x \rangle$ is a strict Lyapunov function for $\dot{x} = Ax + h(x)Mx$ (i.e. $\dot{V} < 0$ on $\mathbb{R}^n \setminus \{0\}$);
- (b) $PM + M^\top P = 0$ (equivalently, $M = P^{-1}S$ with S skew).

In that case, $\dot{V} = -\langle Q_P x, x \rangle$ for every h , and the origin is globally exponentially stable for every locally Lipschitz h .

Proof. (b) $\Rightarrow$ (a): $\langle Px, Mx \rangle = \frac{1}{2} \langle (PM + M^\top P)x, x \rangle = 0$, so $\dot{V} = -\langle Q_P x, x \rangle < 0$; Theorem 3 gives the rest. (a) $\Rightarrow$ (b): if $c := \langle Px_0, Mx_0 \rangle \neq 0$ for some $x_0 \neq 0$, choose a Lipschitz h with $h(x_0) = R \text{sign}(c)$; then $\dot{V}(x_0) = -\langle Q_P x_0, x_0 \rangle + 2R|c| > 0$ for R large. Hence $\langle Px, Mx \rangle = 0$ for all x , i.e. the symmetric matrix $PM + M^\top P$ vanishes.

Hence the set of directions certified for all h by *some* quadratic function is

$$\mathcal{M}(A) = \bigcup_{P \in \mathcal{P}(A)} P^{-1} \mathfrak{so}(n),$$

where $\mathfrak{so}(n)$ are skew matrices. The condition “ $\exists P > 0 : PM + M^\top P = 0, A^\top P + PA < 0$ ” is a linear matrix inequality in P , so membership $M \in \mathcal{M}(A)$ is checkable by semidefinite programming.

Proposition 7 (The Euclidean rotation in the plane). *Let $n = 2$ and A Hurwitz. Then $\Omega \in \mathcal{M}(A)$ iff $A + A^\top < 0$. More generally, a skew $J \in \mathbb{R}^{n \times n}$ lies in $\mathcal{M}(A)$ iff there is $P \in \mathcal{P}(A)$ commuting with J .*

Proof. For J skew, $PJ + J^\top P = PJ - JP$, which vanishes iff P commutes with J . For $J = \Omega$ and $n = 2$, matrices commuting with Ω are $\alpha I + \beta \Omega$; symmetry forces $\beta = 0$, so $P = \alpha I$, and $P \in \mathcal{P}(A)$ iff $A + A^\top < 0$.

Example 8 (Directions that are, and are not, certified). *For A of Example 5, $A + A^\top$ has eigenvalue $8 > 0$, so the Euclidean rotation $h\Omega x$ is not certified; indeed for the constant $h = 5$, $A + 5\Omega = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix}$ has eigenvalues $4, -6$ and is unstable. On the other hand*

$$M = P_0^{-1} \Omega = \frac{1}{6.5} \begin{pmatrix} -2.5 & -25.5 \\ 0.5 & 2.5 \end{pmatrix} \in \mathcal{M}(A),$$

so $\dot{x} = Ax + h(x)Mx$ is globally exponentially stable for every locally Lipschitz h , e.g. $h = 50\|x\|^2$, for which f grows like $\|x\|^3$. (Figure 3a,b.)

Definition 1 (Port-Hamiltonian class). For $P > 0$ let $\mathfrak{H}_P$ be the set of locally Lipschitz f with $\langle Px, f(x) \rangle \leq 0$. Equivalently $f = P^{-1}(S(x) - R(x))Px$ with $S(x)$ skew and $R(x) \geq 0$ symmetric, both arbitrary (with $y = Px$, $g = Pf$: $R = -\frac{\langle y, g \rangle}{\|y\|^2} I$, $S = \frac{g_\perp y^\top - y g_\perp^\top}{\|y\|^2}$, $g_\perp = g - \frac{\langle y, g \rangle}{\|y\|^2} y$).

Theorem 9. *Let $P \in \mathcal{P}(A)$ and $f = f_1 + f_2$ with $f_1 \in \mathfrak{H}_P$ and f_2 locally Lipschitz satisfying (2) (e.g. $\|f_2\| \leq \gamma\|x\|$ with $A^\top P + PA + P^2 + \gamma^2 I < 0$). Then the origin of (1) is globally exponentially stable.*

Proof. $2\langle Px, f \rangle \leq 2\langle Px, f_2 \rangle$; apply Theorem 3.

5 Gain–dissipation trade-off, time variation and ISS

Theorem 10. *Let $\gamma_0 \geq 0$ and $P > 0$ with $A^\top P + PA + P^2 + \gamma_0^2 I \leq -\varepsilon I$, $\varepsilon > 0$. Let $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be piecewise continuous in t , locally Lipschitz in x uniformly in t on compacts, $f(t, 0) = 0$, $f = f_1 + f_2$, and suppose there are continuous $g, \gamma : \mathbb{R}^n \rightarrow [0, \infty)$ with*

$$\langle Px, f_1(t, x) \rangle \leq -g(x)\|x\|^2, \quad (\text{T1})$$

$$\|f_2(t, x)\| \leq \gamma(x)\|x\|, \quad (\text{T2})$$

$$\gamma(x)^2 \leq \gamma_0^2 + 2g(x) \quad \forall x. \quad (\text{T3})$$

Then (1) the origin of $\dot{x} = Ax + f(t, x)$ is globally exponentially stable, uniformly in t_0 ; (2) for $\dot{x} = Ax + f(t, x) + d(t)$ with d bounded and piecewise continuous,

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\rho t/2} \|x(0)\| + \sqrt{\frac{4\|P\|^2}{\varepsilon \rho \lambda_{\min}(P)}} \sup_{s \leq t} \|d(s)\|, \quad \rho = \frac{\varepsilon}{2\lambda_{\max}(P)}.$$

Proof. With $V = \langle Px, x \rangle$: $\dot{V} = \langle (A^\top P + PA)x, x \rangle + 2\langle Px, f_1 \rangle + 2\langle Px, f_2 \rangle + 2\langle Px, d \rangle$. By (T1), $2\langle Px, f_1 \rangle \leq -2g\|x\|^2$. By Young's inequality, (T2) and (T3), $2\langle Px, f_2 \rangle \leq \langle P^2 x, x \rangle + (\gamma_0^2 + 2g)\|x\|^2$. Hence $\dot{V} \leq -\varepsilon\|x\|^2 + 2\langle Px, d \rangle$. For $d = 0$ this gives (1). Otherwise $2\langle Px, d \rangle \leq \frac{\varepsilon}{2}\|x\|^2 + \frac{2}{\varepsilon}\|Pd\|^2$, so $\dot{V} \leq -\rho V + \frac{2\|P\|^2}{\varepsilon}\|d\|^2$ and $V(t) \leq e^{-\rho t}V(0) + \frac{2\|P\|^2}{\varepsilon\rho} \sup \|d\|^2$; take square roots. There is no finite escape since V is bounded on bounded time intervals.

Example 11 (Quadratic growth). $\dot{x} = -x - x^3 + cx^2$: take $P = 1$, $\gamma_0 = 0$ ($-2 + 1 < 0$), $f_1 = -x^3$, $g = x^2$, $f_2 = cx^2$, $\gamma(x) = c|x|$. (T3) reads $c^2 \leq 2$, so GAS for $c \leq \sqrt{2}$, while the exact range is $c < 2$ (for $c \geq 2$ the roots of $x^2 - cx + 1 = 0$ are equilibria). The method captures 71% of the optimal interval for a perturbation of quadratic growth, outside any sector condition.

6 Optimality and the gap to the real stability radius

Let $r_{NL}(A)$ be the supremum of γ such that the origin of (1) is GAS for every locally Lipschitz f with $f(0) = 0$ and $\|f(x)\| \leq \gamma\|x\|$ (including γ itself). Proposition 4 with Theorem 10 gives $r_{NL} \geq r_{\mathbb{C}}$. A real Δ with $\|\Delta\| = r_{\mathbb{R}}$ and an eigenvalue on $i\mathbb{R}$ is admissible and not asymptotically stable, so $r_{NL} \leq r_{\mathbb{R}}$. Thus $r_{\mathbb{C}} \leq r_{NL} \leq r_{\mathbb{R}}$. Let $\alpha(A) = \min_j |\operatorname{Re} \lambda_j(A)|$.

Theorem 12. *Let A be Hurwitz.*

$$(0) \quad r_{\mathbb{R}}(A) \leq \alpha(A).$$

$$(a) \quad \text{If } \sup_{\omega} \|(i\omega I - A)^{-1}\| \text{ is attained at } \omega = 0, \text{ then } r_{\mathbb{C}}(A) = r_{\mathbb{R}}(A) \text{ and hence } r_{NL}(A) = r_{\mathbb{C}}(A).$$

$$(b) \quad \text{If } A \text{ is normal, then } r_{\mathbb{C}}(A) = r_{\mathbb{R}}(A) = \alpha(A) \text{ and hence } r_{NL}(A) = \alpha(A).$$

In these cases the quadratic threshold is the exact global threshold; no Lyapunov function of any form improves it.

Proof. (0) $\Delta = \alpha I$ puts an eigenvalue of $A + \Delta$ on $i\mathbb{R}$. (a) $M = (-A)^{-1}$ is real, so its top singular triple (σ, u, v) , $Mv = \sigma u$, is real. Put $\Delta = \sigma^{-1}vu^\top$: $\|\Delta\| = 1/\sigma = r_{\mathbb{C}}$ and $M\Delta u = \sigma^{-1}Mv(u^\top u) = u$, so $\det(-A - \Delta) = 0$. Thus $r_{\mathbb{R}} \leq r_{\mathbb{C}}$. (b) For normal A , $\|(i\omega I - A)^{-1}\| = 1/\min_j |i\omega - \lambda_j|$, with supremum $1/\alpha$ (at $\omega = \operatorname{Im} \lambda_j$); so $r_{\mathbb{C}} = \alpha$; by (0), $\alpha = r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq \alpha$.

Example 13. For A of Example 5 the supremum is attained at $\omega = 0$, so $r_{NL}(A) = r_{\mathbb{C}}(A) = 0.0990195$ exactly: $\Delta = \sigma_{\max}^{-1}vu^\top$ has norm 0.0990195 and $A + \Delta$ has eigenvalues 0 and -1.9806 .

The non-normal case

When the maximizing frequency is not 0 and A is not normal, $r_{\mathbb{C}} < r_{\mathbb{R}}$ can occur: among 100 random Hurwitz 3×3 matrices, 77% had $r_{\mathbb{R}} = r_{\mathbb{C}}$ and the largest ratio $r_{\mathbb{R}}/r_{\mathbb{C}}$ was 1.63 (Figure 2). Take

$$A_3 = \begin{pmatrix} -1.0606 & -0.8753 & -2.2695 \\ -0.6169 & -1.5363 & 2.0971 \\ -0.3728 & -1.3298 & 0.1652 \end{pmatrix}, \quad r_{\mathbb{C}} = 0.1671, \quad r_{\mathbb{R}} = 0.2523.$$

Proposition 14 (Time-varying perturbations below $r_{\mathbb{R}}$). *There is a T -periodic ($T = 11.0522$), piecewise constant, real $\Delta(t)$ with $\sup_t \|\Delta(t)\| \leq 0.2256 < r_{\mathbb{R}}(A_3)$ (with $r_{\mathbb{R}} = 0.2523$ computed numerically) such that $\dot{x} = (A_3 + \Delta(t))x$ is unstable.*

Proof. Let $\Delta(t) = \Delta_k$ on $[kT/8, (k+1)T/8)$, $k = 0, \dots, 7$ (the matrices are listed in the supplementary files). The monodromy matrix is $M = \prod_{k=7}^0 \exp((A_3 + \Delta_k)T/8)$. One finds $\det(I - M) = -0.3038 < 0$ and $\det(3I - M) = 14.708 > 0$. The characteristic polynomial $p(\lambda) = \det(\lambda I - M)$ has $p(1) < 0 < p(3)$, hence a real root in $(1, 3)$: a Floquet multiplier of modulus > 1 . Repeated squaring of $\Delta_k^\top \Delta_k$ gives $\max_k \|\Delta_k\|_2 \leq 0.2255986 < 0.2256$. The numerical value of the multiplier is 1.327.

Definition 2 (Time-varying radius). Call (A, γ) *TV-stable* if there are $C, \mu > 0$ such that every solution of $\dot{x} = (A + \Delta(t))x$, with $\Delta : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ measurable and $\|\Delta(t)\| \leq \gamma$ a.e., satisfies $\|x(t)\| \leq Ce^{-\mu t} \|x(0)\|$. Let $r_{TV}(A) := \sup\{\gamma : (A, \gamma) \text{ is TV-stable}\}$.

Proposition 15. $r_{\mathbb{C}}(A) \leq r_{TV}(A) \leq r_{NL}(A)$.

Proof. The first inequality follows from Theorem 3 (whose proof applies verbatim to time-dependent f) with $f(x, t) = \Delta(t)x$ and Proposition 4. For the last one let $\gamma < r_{TV}(A)$, f admissible with constant γ , $x(0) \neq 0$ and x the maximal solution of (1). By uniqueness (f locally Lipschitz, $f(0) = 0$) $x(t) \neq 0$, and $\|x(t)\| \leq e^{(\|A\| + \gamma)t} \|x(0)\|$ excludes finite escape. Put $\Delta(t) = f(x(t))x(t)^\top / \|x(t)\|^2$: it is continuous, $\|\Delta(t)\| = \|f(x(t))\| / \|x(t)\| \leq \gamma$ and $\Delta(t)x(t) = f(x(t))$, so x solves the linear time-varying system. Hence $\|x(t)\| \leq Ce^{-\mu t} \|x(0)\|$ with C, μ independent of f and $x(0)$, and the origin is globally exponentially stable. Therefore $r_{NL}(A) \geq \gamma$ for every $\gamma < r_{TV}(A)$.

Lemma 16. Let Φ_Δ be the fundamental matrix of $\dot{x} = (A + \Delta(t))x$, $s_\gamma(t) := \sup\{\|\Phi_\Delta(t)\| : \Delta \text{ measurable}, \|\Delta\| \leq \gamma\}$. Then:

- (a) $\lambda_{\max}(\gamma) := \lim_{t \rightarrow \infty} \frac{1}{t} \log s_\gamma(t)$ exists and equals $\inf_{t > 0} \frac{1}{t} \log s_\gamma(t)$; (A, γ) is TV-stable iff $\lambda_{\max}(\gamma) < 0$.
- (b) $\lambda_{\max}(\gamma + \eta) \geq \lambda_{\max}(\gamma) + \eta$ for all $\eta \geq 0$.
- (c) If $\lambda_{\max}(\gamma) < 0$, then $\lambda_{\max}(\gamma') < 0$ for some $\gamma' > \gamma$.

Consequently $r_{TV}(A) = \sup\{\gamma : \lambda_{\max}(\gamma) < 0\}$ and $\lambda_{\max}(\gamma) \geq \gamma - r_{TV}(A)$ for every $\gamma > r_{TV}(A)$.

Proof. (a) The class of admissible Δ is invariant under time shifts and concatenation, and $\Phi_\Delta(t+r) = \Phi_{\Delta(t+\cdot)}(r)\Phi_\Delta(t)$; hence $s_\gamma(t+r) \leq s_\gamma(r)s_\gamma(t)$, and Fekete's lemma gives the limit and the infimum (finite, since $e^{-(\|A\| + \gamma)t} \leq s_\gamma(t) \leq e^{(\|A\| + \gamma)t}$). If $\lambda_{\max} < 0$, there is t_0 with $s_\gamma(t_0) \leq e^{-2\mu t_0}$ for some $\mu > 0$, and $s_\gamma(kt_0 + r) \leq s_\gamma(t_0)^k s_\gamma(r) \leq e^{(\|A\| + \gamma)t_0} e^{-2\mu(kt_0)}$ for $r \in [0, t_0]$, which is TV-stability; the converse is immediate. (b) For $\|\Delta\| \leq \gamma$ the matrix $\Delta + \eta I$ is admissible for $\gamma + \eta$ and $\Phi_{\Delta + \eta I}(t) = e^{\eta t} \Phi_\Delta(t)$, so $s_{\gamma + \eta}(t) \geq e^{\eta t} s_\gamma(t)$. (c) Let $\|\Phi_\Delta(t)\| \leq Ce^{-\mu t}$ for all admissible Δ ($\|\Delta\| \leq \gamma$) and put $V(x) = \sup_{\Delta, t \geq 0} e^{\mu t/2} \|\Phi_\Delta(t)x\|$. Then V is a norm, $\|x\| \leq V(x) \leq C\|x\|$, and by concatenation $V(\Phi_\Delta(h)x) \leq e^{-\mu h/2} V(x)$. Let $\|\Delta'\| \leq \gamma + \varepsilon$ and write $\Delta' = \Delta_1 + \Delta_2$ with $\Delta_1 = \frac{\gamma}{\gamma + \varepsilon} \Delta'$ ($\|\Delta_1\| \leq \gamma$, $\|\Delta_2\| \leq \varepsilon$). For a solution x of $\dot{x} = (A + \Delta')x$, the variation of constants formula with forcing $\Delta_2 x$ gives, with $K = \|A\| + \gamma + \varepsilon$,

$$V(x(t+1)) \leq e^{-\mu/2} V(x(t)) + \int_t^{t+1} V(\Delta_2(s)x(s)) ds \leq (e^{-\mu/2} + C\varepsilon e^K) V(x(t)).$$

For ε small the factor is $\leq e^{-\mu/4} < 1$ uniformly in Δ' , hence $(A, \gamma + \varepsilon)$ is TV-stable. The last statement follows from (a), (b): for $\gamma > \gamma_0 > r_{TV}$, $\lambda_{\max}(\gamma_0) \geq 0$, so $\lambda_{\max}(\gamma) \geq \gamma - \gamma_0$.

Theorem 17 ($r_{NL} = r_{TV}$ for $n \geq 3$). Let $A \in \mathbb{R}^{n \times n}$ be Hurwitz with $n \geq 3$. Then $r_{NL}(A) = r_{TV}(A)$. More precisely, for every $\gamma > r_{TV}(A)$ there is a locally Lipschitz $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, vanishing near 0, with $\|f(x)\| \leq \gamma\|x\|$ and such that $\dot{x} = Ax + f(x)$ has an unbounded solution.

Proof. $r_{NL} \geq r_{TV}$ is Proposition 15. For the converse let $\gamma > r_{TV}$, $\kappa := (\gamma - r_{TV})/4$, $\gamma_1 := r_{TV} + 2\kappa$. By Lemma 16, $\lambda_{\max}(\gamma_1) \geq 2\kappa > 0$.

(i) *A periodic destabilizing Δ .* Let $\Sigma = \{\Phi_\Delta(1) : \|\Delta\| \leq \gamma_1\}$, a bounded set of matrices. Products of k elements of Σ are the matrices $\Phi_\Delta(k)$ of concatenated Δ 's, so the joint spectral radius of Σ is $e^{\lambda_{\max}(\gamma_1)} > 1$. By the Berger–Wang theorem [23] it equals $\limsup_k \sup\{\rho(S_1 \cdots S_k)^{1/k}\}$, hence there are k and $S_i \in \Sigma$ with $\rho(P) > 1$, $P = S_k \cdots S_1$. Let Δ_0 be the k -periodic measurable extension to $\mathbb{R}$ of the corresponding concatenation, $\|\Delta_0\| \leq \gamma_1$, and μ an eigenvalue of P with $|\mu| = \rho(P)$, with eigenvector $w = a + ib$. If $\mu \in \mathbb{R}$ take $v = a$ (or b) $\neq 0$; otherwise a, b are independent and we take $v = a$, so that $\|P^m v\| = |\mu|^m \|\cos(m\theta)a - \sin(m\theta)b\| \geq |\mu|^m \sigma_{\min}([a \ b])$. In

both cases the solution x_0 of $\dot{x} = (A + \Delta_0)x$, $x_0(0) = v$, satisfies $\|x_0(t)\| \rightarrow \infty$ as $t \rightarrow \infty$ (on $[mk, (m+1)k]$ it is comparable with $\|x_0(mk)\|$ up to the factor $e^{(\|A\| + \gamma_1)k}$), and $x_0 \neq 0$.

(ii) *A smooth admissible curve.* Let $x_1 = x_0 * \varphi_\varepsilon$ with a mollifier φ_ε supported in $[-\varepsilon, \varepsilon]$. Then x_1 is C^∞ and

$$\dot{x}_1(t) - Ax_1(t) = \int \varphi_\varepsilon(\tau) \Delta_0(t - \tau) x_0(t - \tau) d\tau, \quad \text{so} \quad \|\dot{x}_1 - Ax_1\| \leq \gamma_1 \max_{|\tau| \leq \varepsilon} \|x_0(t - \tau)\| \leq (\gamma_1 + \frac{\kappa}{2}) \|x_1(t)\|$$

for ε small, because $\|x_0(t - \tau) - x_0(t)\| \leq (\|A\| + \gamma_1) \varepsilon e^{(\|A\| + \gamma_1)\varepsilon} \|x_0(t)\|$. Also $\|x_1(t)\| \rightarrow \infty$.

(iii) *General position.* The map $x_1 : (-1, \infty) \rightarrow \mathbb{R}^n$ is smooth and $n \geq 2 \cdot 1 + 1$. By Whitney's theorem, injective immersions are dense in $C^\infty((-1, \infty), \mathbb{R}^n)$ for the strong C^1 topology (see e.g. [24], Ch. 2), so there is an injective immersion x with $\|x(t) - x_1(t)\| + \|\dot{x}(t) - \dot{x}_1(t)\| \leq \varepsilon' \|x_1(t)\|$ for all t , where $\varepsilon' > 0$ is chosen with $(\gamma_1 + \frac{\kappa}{2} + \varepsilon'(1 + \|A\|))/(1 - \varepsilon') \leq \gamma_1 + \kappa$. Then

$$\|\dot{x}(t) - Ax(t)\| \leq (\gamma_1 + \kappa) \|x(t)\|, \quad \|x(t)\| \geq (1 - \varepsilon') \|x_1(t)\| \rightarrow \infty.$$

Hence $x|_{[0, \infty)}$ is a proper injective immersion with $x \neq 0$: $\Gamma := x([0, \infty))$ is a closed set not containing a neighbourhood of 0, and $\Gamma \cap \bar{B}_R = x([0, T_R])$ is compact for every R .

(iv) *The state feedback.* For $y = x(t) \in \Gamma$ put $F(y) := \dot{x}(t) - Ax(t)$; this is well defined because x is injective, and $\|F(y)\| \leq (\gamma_1 + \kappa) \|y\|$. For $J = [0, T_R]$, injectivity and $\dot{x} \neq 0$ give L_R with $|t - s| \leq L_R \|x(t) - x(s)\|$ on J (otherwise a sequence t_j, s_j with $\|x(t_j) - x(s_j)\|/|t_j - s_j| \rightarrow 0$ would converge to $t_* \neq s_*$ with $x(t_*) = x(s_*)$, or to $t_* = s_*$ with $\dot{x}(t_*) = 0$); as $F \circ x = \dot{x} - Ax$ is Lipschitz on J , F is Lipschitz on $\Gamma \cap \bar{B}_R$ for every R . Extend each restriction $F|_{\Gamma \cap \bar{B}_{m+1}}$ to $\mathbb{R}^n$ with the same Lipschitz constant (McShane [25], componentwise), glue the extensions with a locally finite Lipschitz partition of unity $\{\psi_m\}$ subordinate to $\{B_{m+1}\}$ to get a locally Lipschitz $\tilde{F}$ with $\tilde{F} = F$ on Γ , and set

$$f(y) = \chi(\|y\|) \Pi_{\gamma\|y\|}(\tilde{F}(y)),$$

where Π_r is the radial projection onto the ball of radius r (it is 1-Lipschitz in w and in r) and χ is a Lipschitz cutoff with $\chi = 0$ near 0 and $\chi = 1$ on $\{\|y\| \geq \min_\Gamma \|y\|\}$. Then f is locally Lipschitz, $f = 0$ near 0, $\|f(y)\| \leq \gamma \|y\|$, and $f = F$ on Γ because $\|F\| \leq (\gamma_1 + \kappa) \|y\| < \gamma \|y\|$ there. Therefore $\dot{x}(t) = Ax(t) + f(x(t))$, and this solution is unbounded. Thus the origin is not globally asymptotically stable for this f , and $r_{NL}(A) \leq \gamma$. Since $\gamma > r_{TV}$ was arbitrary, $r_{NL} \leq r_{TV}$.

Remark 1. The dimension enters only through Whitney's theorem (injective immersions of a curve are generic iff $n \geq 3$). For $n = 2$ a planar curve that winds around the origin cannot in general be perturbed into an injective one, and the question whether $r_{NL} = r_{TV}$ stays open here. Theorem 19 below is the explicit, quantitative counterpart of Theorem 17 for the example A_3 (it avoids general position by checking injectivity of one self-similar spiral with certified margins).

Theorem 18 (When is the quadratic threshold exact?). *Let A be Hurwitz.*

(a) *If $\lambda_{\max}(r_{\mathbb{C}}(A)) < 0$, i.e. $(A, r_{\mathbb{C}}(A))$ is TV-stable, then $r_{NL}(A) \geq r_{TV}(A) > r_{\mathbb{C}}(A)$ (any n).*

(b) *If $\lambda_{\max}(r_{\mathbb{C}}(A)) \geq 0$, then $r_{TV}(A) = r_{\mathbb{C}}(A)$, and for $n \geq 3$ also $r_{NL}(A) = r_{\mathbb{C}}(A)$.*

Hence, for $n \geq 3$: $r_{NL}(A) = r_{\mathbb{C}}(A)$ iff the differential inclusion $\dot{x} \in (A + r_{\mathbb{C}}(A)\mathbb{B})x$ is not uniformly exponentially stable. This holds, in particular, when A is normal or $\omega^ = 0$ (Theorem 12), where a constant real Δ with $\|\Delta\| = r_{\mathbb{C}}$ puts an eigenvalue of $A + \Delta$ on $i\mathbb{R}$.*

Proof. $r_{\mathbb{C}} \leq r_{TV} \leq r_{NL}$ by Proposition 15. (a) By Lemma 16(c) there is $\gamma' > r_{\mathbb{C}}$ with (A, γ') TV-stable, so $r_{TV} \geq \gamma' > r_{\mathbb{C}}$. (b) For $\gamma > r_{\mathbb{C}}$, Lemma 16(b) gives $\lambda_{\max}(\gamma) \geq \lambda_{\max}(r_{\mathbb{C}}) + (\gamma - r_{\mathbb{C}}) > 0$, so (A, γ) is not TV-stable; hence $r_{TV} \leq r_{\mathbb{C}}$. For $n \geq 3$ apply Theorem 17.

Theorem 19 (Time-invariant perturbations below $r_{\mathbb{R}}$). *There is a locally Lipschitz, time-invariant $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $f(0) = 0$ and $\|f(x)\| \leq 0.2019\|x\| < r_{\mathbb{R}}(A_3)$ such that $\dot{x} = A_3x + f(x)$ has an unbounded solution. Hence $0.1671 \leq r_{NL}(A_3) \leq 0.2019 < r_{\mathbb{R}}(A_3)$. Moreover Step 1 below proves $r_{TV}(A_3) \leq 0.2018005$, which by Theorem 17 already gives $r_{NL}(A_3) \leq 0.2018005$ without Steps 2–4 (Proposition 20 improves this to 0.2000007); Steps 2–4 construct an explicit f with constant 0.2019 and a quantitative embedding margin.*

Proof. Data. Let $T = 2.751226$, $t_k = kT/16$, and let $B(t)$ be the T -periodic continuous piecewise linear matrix function with $B(t_k) = B_k$, $k = 0, \dots, 15$, where the B_k are the 3×3 matrices with 6 decimals listed in the supplementary file `knots16.txt`. Convexity gives $\|B(t)\| \leq \max_k \|B_k\|_2$, and (repeated squaring of $B_k^\top B_k$) $\max_k \|B_k\|_2 \leq 0.2018005 < 0.2019$. Put $C(t) = A_3 + B(t)$, $K_* := \|A_3\|_2 + 0.2019 = 3.366$, $\sigma := 2 \cdot 0.2019/(T/16) = 2.35 \geq \|\dot{B}\|$.

Step 1 (monodromy). The monodromy matrix M is the product of the fundamental matrices over the 256 steps $[t_k + jh, t_k + (j+1)h]$, $h = T/256$, and, for $p(\lambda) = \det(\lambda I - M)$,

$$p(-1) = 0.00271195 > 0, \quad p(-3) = -15.2243 < 0.$$

Since p is a real polynomial, M has a real eigenvalue $\lambda \in (-3, -1)$ and a real eigenvector v . (Numerically $\lambda = -1.00500$, other multipliers -0.4606 and 0.0051 .) Let x_p solve $\dot{x} = C(t)x$, $x_p(0) = v$; then $x_p(t+T) = \lambda x_p(t)$ and $\|x_p(kT)\| = |\lambda|^k \|v\| \rightarrow \infty$. This already shows $r_{TV}(A_3) \leq 0.2018005$.

Step 2 (embedding, a finite check). Extend x_p to $t \in \mathbb{R}$ by $x_p(t+T) = \lambda x_p(t)$. Let $\Gamma = \{x_p(t)\}$.

(L) *Local injectivity.* Let $m_i := \|\dot{x}_p(t_i)\|/\|x_p(t_i)\|$ at a grid node. For $|t - t_i| \leq \eta' := 0.0263$, $\|\ddot{x}_p\| \leq (\sigma + K_*^2)\|x_p\|$ and $\|x_p(t)\| \leq e^{K_*\eta'}\|x_p(t_i)\|$, so $\|\dot{x}_p(t) - \dot{x}_p(t_i)\| \leq 0.394\|x_p(t_i)\| < m_i\|x_p(t_i)\|$ as soon as $m_i > 0.394$. Then $\langle \dot{x}_p(t), \dot{x}_p(t_i) \rangle > 0$ on the window, hence $\langle x_p(t) - x_p(s), \dot{x}_p(t_i) \rangle > 0$ for $s < t$ in it, and $x_p(t) \neq x_p(s)$ whenever $0 < |t - s| \leq 0.0526 - \delta$ (take the node nearest to $(t+s)/2$, so both t, s lie within η' of it). On the $1.6 \cdot 10^4$ nodes $\min m_i \geq 0.4145$.

(N) *Non-local separation.* Let $\tau_i = iT/n$, $n = 16000$, $\delta = T/n = 1.72 \cdot 10^{-4}$. Comparing $x_p(\tau_i)$ with $\lambda^k x_p(\tau_j)$ over all $|k| \leq 382$ and all pairs with $|i - j - kn| \geq 303$ gives the smallest relative distance 0.0062282 , so $c_{\text{grid}} \geq 0.0062$. (Since $306\delta \geq 0.0526$, every pair of true times with $|t - s| \geq 0.0526 - 2\delta$ has node indices with $|i - j - kn| \geq 303$. The k -range suffices: $\|x_p\|$ varies by a factor ≤ 2.6659 over a period, $|\lambda|^{382} \geq 6.726$ and $|\lambda|^{-382} \leq 0.1487$, so every pair with $k > 382$ (resp. $k < -382$) has relative distance ≥ 1.52 (resp. ≥ 0.60).) Since $\|\dot{x}_p\| \leq K_*\|x_p\|$, moving each argument by at most δ changes the relative distance by at most $3(e^{K_*\delta} - 1) \leq 0.0017369$; the pairs of true times with $|t - s| \geq 0.0526 - 2\delta$ are thus covered, and with (L) all $t \neq s$. Hence

$$c := \inf \left\{ \frac{\|x_p(t) - x_p(s)\|}{\|x_p(t)\|} : |t - s| \geq 0.0526 - 2\delta \right\} \geq (0.0062 - 0.0017369)e^{-K_*\delta} \geq 0.0044605.$$

Together, x_p is injective on $\mathbb{R}$, i.e. Γ is an embedded $C^{1,1}$ curve.

Step 3. Local curvature: $\kappa\|x_p\| \leq (\sigma + K_*^2)/m^2 \leq 80$, so the tube of relative radius $1/80 > c/4$ is a tubular neighbourhood. Let $\varepsilon = c/4 = 0.0011$; for $x \neq 0$ let $\tau(x)$ and $r(x) = \text{dist}(x, \Gamma)/\|x\|$ be the phase and relative distance of the nearest point of Γ (well defined for $r < c/2$), and

$$f(x) = \chi(r(x)/\varepsilon) B(\tau(x))x,$$

with $\chi = 1$ on $[0, \frac{1}{2}]$, $\chi = 0$ on $[1, \infty)$ and a C^1 transition. Then $\|f\| \leq 0.2019\|x\|$, and f is locally Lipschitz (projection onto an embedded $C^{1,1}$ curve is Lipschitz in a tube, B is Lipschitz in time, and the tube scales with $\|x\|$).

Step 4. On Γ , $r = 0$, $\chi = 1$, $\tau(x_p(t)) = t$, so $f(x_p(t)) = B(t)x_p(t)$ and x_p solves $\dot{x} = A_3x + f(x)$; since $x_p(kT) = \lambda^k x_p(0)$ with $|\lambda| > 1$, it is unbounded.

Remark 2. (i) The growing solution lies on an unstable invariant spiral, which does not matter: one unbounded solution refutes global asymptotic stability (Figure 4). (ii) The bound 0.2019 improves the values 0.2181 and 0.2507 of the previous versions. Re-optimising the knots with 24 instead of 16 (seeded by resampling) gives a positive exponent $+9 \cdot 10^{-5}$ at $\gamma = 0.200$, see Proposition 20. (iii) For rank-one perturbations the failure of Aizerman- and Kalman-type conjectures [17, 18] also gives time-invariant destabilization below the linear margin; here the perturbation class is the unstructured sector. (iv) In the sense of [16], the nonlinear radius is bounded below by $r_{\mathbb{C}}$ of the linearisation; the example shows the bound is not attained by the real radius.

Proposition 20 (A sharper periodic destabilization, 24 knots). *Let $T = 2.750603$, $t_k = kT/24$, and let $B(t)$ be the T -periodic continuous piecewise linear function with $B(t_k) = B_k$, $k = 0, \dots, 23$, given with 6 decimals in the supplementary file `knots24.txt`. Then $\|B(t)\| \leq 0.2000007$ and the monodromy matrix M of $\dot{x} = (A_3 + B(t))x$ has a real eigenvalue in $(-3, -1)$, namely $\mu = -1.00024428\dots$. Hence $r_{TV}(A_3) \leq 0.2000007$ and, by Theorem 17, $r_{NL}(A_3) \leq 0.2000007 < r_{\mathbb{R}}(A_3)$.*

Proof. Exactly as in Step 1 of Theorem 19, with 264 steps of length $T/264$: $\max_k \|B_k\|_2 \leq 0.2000006545$ (repeated squaring), and

$$p(-1) = 0.000131953 > 0, \quad p(-3) = -15.2486 < 0,$$

so M has a real eigenvalue in $(-3, -1)$, i.e. $|\mu| > 1$ (it equals -1.000244284535583 ; the other two multipliers are 0.005125 and -0.462592). Since μ is real with $|\mu| > 1$ the solution with $x(0)$ the real eigenvector grows without bound, so $(A_3, 0.2000007)$ is not TV-stable, which gives the bound on r_{TV} ; Theorem 17 ($n = 3$) gives the bound on r_{NL} .

Remark 3. Here $|\mu| - 1 \approx 2.4 \cdot 10^{-4}$, so the spiral of the corresponding solution would have a relative separation between consecutive turns of order $|\mu|^2 - 1 \approx 5 \cdot 10^{-4}$, which would make the finite embedding check of Step 2 (whose cost grows like $(|\mu| - 1)^{-1}$ in both the number of turns and the grid size, hence like $(|\mu| - 1)^{-2}$ in the number of points) impractical. Theorem 17 removes the need for it. With 24 knots and $\gamma = 0.199$ the best exponent we found is $-1.4 \cdot 10^{-3}$ (one start), so $r_{TV}(A_3) \approx 0.1999$ – 0.2000 appears close to what this family can give.

Consequently: for normal A or maximizer $\omega^* = 0$ the quadratic threshold is exact (Theorem 12); otherwise $r_{NL} < r_{\mathbb{R}}$ can happen and the exact value of r_{NL} is not determined here (for A_3 it lies in $[0.1671, 0.2001]$ by Proposition 20); a non-quadratic V can gain at most a factor 1.20 over $r_{\mathbb{C}}$ for A_3 .

7 Extensions: averaged gain and switching

Theorem 10 applies Young's inequality with a fixed weight, which is conservative when the gain is large only part of the time. We replace it by the *exact* decay rate of V under a sector of gain γ . For $P > 0$ and $\gamma \geq 0$ let

$$\mu_P(\gamma) := \sup \left\{ \mu \in \mathbb{R} : \exists \tau > 0, A^\top P + PA + \mu P + \tau \gamma^2 I + \tau^{-1} P^2 \leq 0 \right\} \in [-\infty, \infty).$$

μ_P is nonincreasing, and $\mu_P(0) > 0$ iff $P \in \mathcal{P}(A)$.

Lemma 21 (Exact rate under a sector). *If $\|w\| \leq \gamma\|x\|$ and $\mu < \mu_P(\gamma)$, then $2\langle Px, Ax + w \rangle \leq -\mu\langle Px, x \rangle$. Conversely, if this holds for all such x, w , then $\mu \leq \mu_P(\gamma)$.*

Proof. For the first claim, $2\langle Px, w \rangle \leq \tau^{-1}\|Px\|^2 + \tau\|w\|^2 \leq \tau^{-1}\langle P^2x, x \rangle + \tau\gamma^2\|x\|^2$ and the defining inequality. The converse follows as in Proposition 4 (S-lemma with the linear maps $f(y) = \langle y, x \rangle w / \|x\|^2$) when $\gamma > 0$.

Theorem 22 (Averaged gain, exact rates). *Let $P > 0$ and $f(t, x)$ with $\|f(t, x)\| \leq \gamma(t)\|x\|$, $\gamma \geq 0$ piecewise continuous and bounded, f locally Lipschitz in x . Then for every solution, $V(x(t)) \leq \exp(-\int_s^t \mu_P(\gamma(r)) dr) V(x(s))$ (for every rate below μ_P). If there are $a \geq 0$, $\nu > 0$ with $\int_s^t \mu_P(\gamma(r)) dr \geq \nu(t-s) - a$ for all $s \leq t$, the origin is globally exponentially stable, with $\|x(t)\| \leq \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)} e^{a/2} e^{-\nu(t-s)/2} \|x(s)\|$.*

Proof. Lemma 21 with $w = f(t, x)$ gives $\dot{V} \leq -\mu_P(\gamma(t))V$ for a.e. t ; apply Gronwall. $\mu_P(\gamma)$ may be negative (unstable bursts).

Theorem 22 replaces the version of the previous draft, which used $e = \gamma^2 - \gamma_0^2 - 2g$ and the factor $\lambda_{\min}(P)/\lambda_{\max}(P)$; for $g = 0$ that version corresponds to the cruder rate $\varepsilon/\lambda_{\max}(P) - \gamma^2/\lambda_{\min}(P) \leq \mu_P(\gamma)$.

Example 23 (Bursts). (a) $n = 1$, $A = -1$, $P = 1$: $\mu_P(\gamma) = 2(1 - \gamma)$. With $\gamma = 3$ during a fraction ϕ of each period and 0 otherwise, $\phi\mu_P(3) + (1 - \phi)\mu_P(0) = 2 - 6\phi > 0$ iff $\phi < 1/3$, which is the exact condition (the previous theorem gave $\phi < 1/9$).

(b) $n = 2$, A of Example 5, $\gamma = 3r_{\mathbb{C}} = 0.2971$. Over P (a 3-parameter family) the best certified duty fraction is $\phi < 0.33333$, with $\mu_P(0) = 1.0098$, $\mu_P(\gamma) = -2.0196$; the previous theorem certifies only $\phi < 1.5 \cdot 10^{-3}$. Conversely, $f = \gamma(t)\Theta x$ with Θ the rank-one matrix $vu^\top$ of Theorem 12(a) and fast switching destabilises at $\phi = r_{\mathbb{C}}/\gamma = 1/3$ (numerical search over Θ and the period found no smaller ϕ : 0.33333). Hence in this example the new bound is sharp. We do not claim this sharpness in general: it relies on $\omega^* = 0$.

Theorem 24 (Switching and averaged rates between certificates). Let $P_i > 0$, $i = 1, \dots, m$, and suppose that in mode i one has $\dot{V}_i \leq -\rho_i V_i$ with $\rho_i \in \mathbb{R}$ (any sign; e.g. $\rho_i = \mu_{P_i}(\gamma_i)$ or $\rho_i = \lambda_{\min}(P_i^{-1/2} M_i P_i^{-1/2})$, $M_i = -(A^\top P_i + P_i A + P_i^2 + \gamma_{0,i}^2 I)$ under (T1)–(T3)). Put

$$\mu := \max \left\{ 1, \max_{i,j} \lambda_{\max}(P_i^{-1} P_j) \right\} \quad \left(= \max_{i,j} \max_{x \neq 0} V_j(x)/V_i(x) \right).$$

Let $\sigma(t)$ be piecewise constant with $N_\sigma(s, t) \leq N_0 + (t - s)/\tau_a$, and suppose the time $T_i(s, t)$ spent in mode i satisfies $\sum_i \rho_i T_i(s, t) \geq \bar{\rho}(t - s) - c$. If $\bar{\rho} > \ln \mu / \tau_a$, the origin is globally exponentially stable.

Proof. Between switches V_σ evolves by $e^{-\rho_i T_i}$; at a switch $V_j \leq \mu V_i$. Hence $V_{\sigma(t)}(x(t)) \leq \mu^{N_0} e^c \exp(-(\bar{\rho} - \ln \mu / \tau_a)(t - s)) V_{\sigma(s)}(x(s))$ [27], and V_i are equivalent to $\|x\|^2$.

Compared with the previous version, three losses are removed: the jump factor is the generalized eigenvalue $\lambda_{\max}(P_i^{-1} P_j)$ instead of $\lambda_{\max}(P_i)/\lambda_{\min}(P_j)$ (these differ by up to $\text{cond}(P)$), the rates are the exact V -relative rates instead of $\varepsilon_i/\lambda_{\max}(P_i)$, and modes with $\rho_i < 0$ are allowed provided the average rate is positive, which unifies Theorems 22 and 24. We have not tabulated a numerical comparison for switching beyond Example 23.

Remark 4 (Non-quadratic certificates for rank-one perturbations). For $f = -b\phi(c^\top x)$ with $0 \leq \sigma\phi(\sigma) \leq k\sigma^2$, the Popov criterion with $V = \langle Px, x \rangle + 2\eta \int_0^{c^\top x} \phi$ [1, Sec. 7.1] gives a threshold never below the circle (quadratic) one; see Table 2.

8 Numerical verification

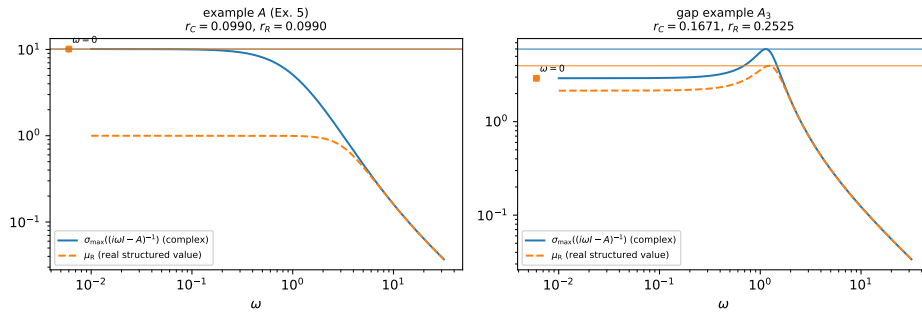
The bound $r_{\mathbb{C}}(A_3) \geq 0.1671$ holds: P is the stabilising solution of $A_3^\top P + PA_3 + P^2 + (0.1671^2 + 5 \cdot 10^{-7})I = 0$, and the Sylvester minors of $-(A_3^\top P + PA_3 + P^2 + 0.1671^2 I)$ and of P are positive; the numerical value is $r_{\mathbb{C}}(A_3) = 0.167104$ at $\omega^* = 1.13605$. The bound $r_{\mathbb{R}}(A_3) > 0.2023$ (hence $0.2019 < r_{\mathbb{R}}$) follows from Qiu's formula $\mu_{\mathbb{R}}(M) = \inf_{g \in (0,1]} \sigma_2 \begin{pmatrix} \text{Re} M & -g \text{Im} M \\ g^{-1} \text{Im} M & \text{Re} M \end{pmatrix}$ [7] with $M(\omega) = (i\omega I - A_3)^{-1}$: on each of 181 cells $[\omega_a, \omega_b] \subset [0, 3.376]$ we fix g and bound σ_2 by its value at the centre plus $(1 + \max(g, 1/g)) \|M\|_{\max}^2 \Delta\omega/2$, using $\|M(\omega)\| \leq 1/0.1671$ (shown above) and Weyl's inequality; for $\omega \geq 3.376$, $\|M\| \leq 4.71 < 1/0.2023$; the largest certified bound on $\mu_{\mathbb{R}}$ is $4.9414 < 1/0.2023$. The numerical value is $\sup_\omega \mu_{\mathbb{R}} = 3.9635$ ($r_{\mathbb{R}} = 0.2523$ at $\omega = 1.2225$).

9 Possible applications

The statements below are candidate uses, not results; the first and second are supported by the theorems above, the others need a model-specific check of the hypotheses.

Table 1 Thresholds γ for $\|f(x)\| \leq \gamma\|x\|$ (classical: $1/(2\lambda_{\max}(P_0))$).

Matrix	classical	$r_{\mathbb{C}}$	$r_{\mathbb{R}}$	r_{NL}	gain vs. classical
A (Ex. 5)	0.0194	0.09902	0.09902	$= 0.09902$ (exact)	$\times 5.1$
A_3	0.0528	0.1671	0.2523	$\in [0.1671, 0.2001]$	$\geq \times 3.2$
normal 4×4 (3 random)	—	$\alpha(A)$	$\alpha(A)$	$= \alpha(A)$ (exact)	—

**Fig. 1** $\sigma_{\max}((i\omega I - A)^{-1})$ (complex) and real structured value $\mu_{\mathbb{R}}$: running example (maximum at $\omega = 0$, so $r_{\mathbb{C}} = r_{\mathbb{R}}$) and A_3 ($\omega^* = 1.137$, $r_{\mathbb{C}} = 0.1671 < r_{\mathbb{R}} = 0.2523$).**Table 2** Verification summary.

Result	Test	Outcome	Status
Prop. 4	$\max_P$ ball threshold vs. $r_{\mathbb{C}}$	0.099027 vs. 0.099020	verified (grid error)
Thm. 6, Ex. 8	V decay for $h = 0, 50\ x\ ^2, 100$	$\dot{V} = -\ x\ ^2$	verified
Thm. 12(a)	random 3×3 with $\omega^* = 0$ (52/60, 77/100)	$r_{\mathbb{R}} = r_{\mathbb{C}} (\leq 10^{-3})$	verified
Thm. 12(b)	3 normal 4×4	$r_{\mathbb{C}} = r_{\mathbb{R}} = \alpha$	verified
Prop. 14	periodic $\Delta(t)$, $1.35 r_{\mathbb{C}}$	$\det(I - M) = -0.304 < 0$	verified
Thm. 19, Step 1	monodromy of $A_3 + B(t)$, 16 knots, $\gamma = 0.2018005$	$p(-1) = 0.0027 > 0 > p(-3)$, $ \lambda > 1$	verified
Prop. 20	monodromy of $A_3 + B(t)$, 24 knots, $\gamma = 0.2000007$	$p(-1) = 1.3 \cdot 10^{-4} > 0 > p(-3)$, $ \mu > 1.00024$	verified
$r_{\mathbb{C}}(A_3) \geq 0.1671$	Riccati inequality, P to 60 digits	Sylvester minors > 0	verified
Thm. 19, Step 2	embedding: local test + grid ($n = 1.6 \cdot 10^4$) with Lipschitz loss	$c \geq 0.0044$, $\min m_i \geq 0.4145 > 0.394$	verified
$r_{\mathbb{R}}(A_3) > 0.2023$	Qiu's formula for $\mu_{\mathbb{R}}$, 181 cells in ω , Lipschitz bounds	$\sup \mu_{\mathbb{R}} \leq 4.9414 < 1/0.2023$	verified
Thm. 22	bursts $3 r_{\mathbb{C}}$, $n = 2$ (Ex. 23)	certifies 0.3333 (old: $1.5 \cdot 10^{-3}$); destabilising 0.3333	verified, sharp here
Thm. 24	proof only (no new numerics)	—	proved
Popov remark	$k = 1.177$: circle 0.774, Popov 1.580	decay to 10^{-23}	verified

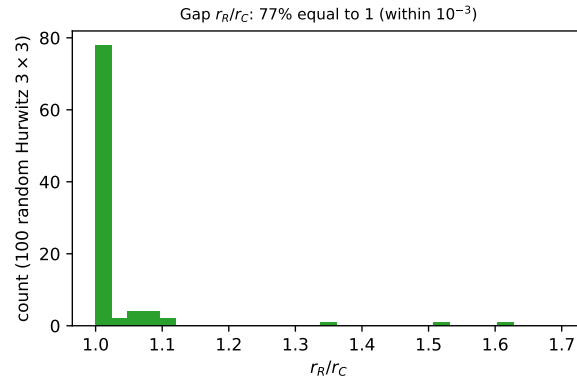


Fig. 2 Distribution of $r_{\mathbb{R}}/r_{\mathbb{C}}$ over 100 random Hurwitz 3×3 matrices: 77% equal 1; maximum 1.63.

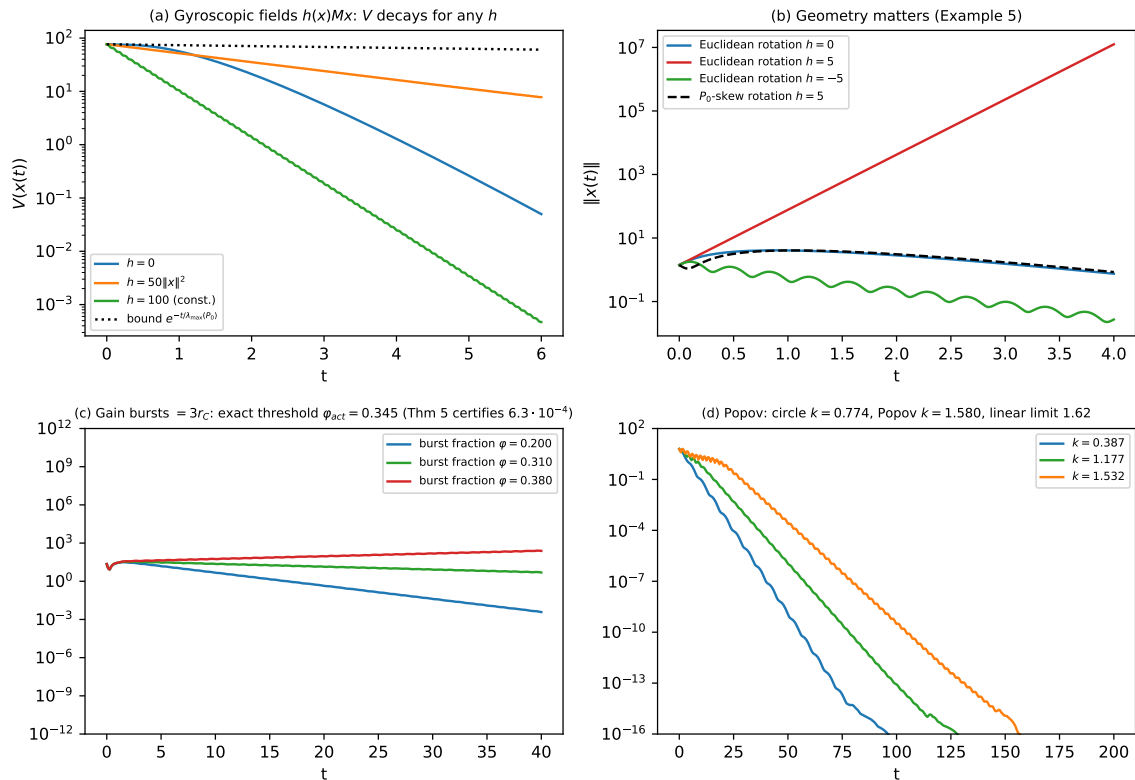


Fig. 3 (a) V for $h(x)Mx$ (Ex. 8). (b) Euclidean rotation destabilizes, P_0 -skew rotation does not. (c) Gain bursts (Thm. 22). (d) Popov-certified rank-one perturbations.

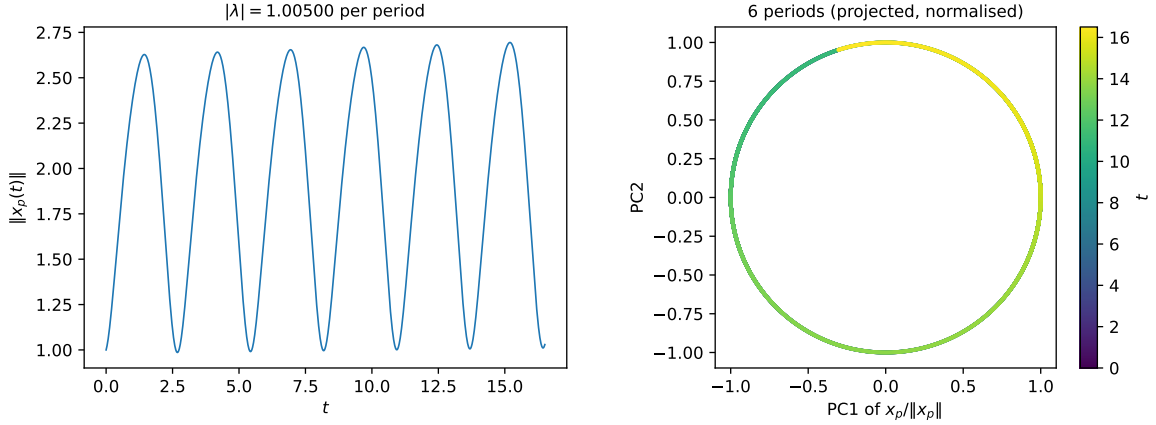


Fig. 4 Theorem 19: norm of the periodic-type solution x_p of $\dot{x} = (A_3 + B(t))x$ with $\|B\| \leq 0.2019 < r_{\mathbb{R}}$ (left; it is the solution of the time-invariant system, Step 4; the multiplier is $\lambda = -1.00500$) and the embedded spiral over six periods, projected on the two leading directions of $x_p/\|x_p\|$ (right).

Neural-network and learned components in feedback loops. If a controller or a learned residual $f(x)$ has a known Lipschitz or sector bound γ , Proposition 4 certifies global stability of $\dot{x} = Ax + f(x)$ whenever $\gamma < r_{\mathbb{C}}(A)$, checked by one frequency sweep or one LMI, and gives a quantified margin over the textbook Lyapunov bound (factor 5.1 in Example 5). This is the setting of recent work on Lur’e systems with neural-network feedback and sector bounds [26]. Our Theorem 19 also shows that, for non-normal A , the real stability radius cannot be used as a global guarantee against nonlinear perturbations.

Gyroscopic and Coriolis terms in mechanical systems. Coriolis and gyroscopic forces do no work: in energy coordinates they are P -skew for the energy matrix P . By Theorem 6, if $P \in \mathcal{P}(A)$ (which requires damping that makes the energy decay strictly), then a coupling $h(x)Mx$ with M P -skew is harmless for arbitrary h , e.g. any speed-dependent gyroscopic gain in rotating machinery, spacecraft or manipulators. The practical content is that the check is a single LMI in P ; whether $P \in \mathcal{P}(A)$ holds for a given machine depends on its damping structure.

Lur’e systems, saturations, phase-locked loops. Sector-bounded nonlinearities (saturation, dead zone, the sin nonlinearity of a PLL) are the classical Aizerman/Kalman setting [17–19]. Our results quantify when a quadratic certificate is exact ($\omega^* = 0$ or normal A) and when hidden destabilization below the real radius can occur.

Economics. Linearised, stable macroeconomic or inventory models with bounded nonlinear adjustments (e.g. a floor on policy rates) have the form $\dot{x} = Ax + f(x)$ with f in a sector. Most such models are in discrete time, so Proposition 4 would have to be restated through the discrete bounded real lemma first. We have not done this.

Intermittent disturbances. Theorem 22 gives a certified fraction of time in which the gain may exceed $r_{\mathbb{C}}$ (communication dropouts, actuator saturation episodes, switched controllers), and in Example 23 this fraction is sharp.

10 Conclusions

We determined what quadratic Lyapunov functions can certify for $\dot{x} = Ax + f(x)$.

1. A quadratic V is a strict certificate iff f lies in a pointwise half-space (Theorem 3); the largest certified sector is $\gamma < r_{\mathbb{C}}(A)$, which improves the classical bound by a factor 5.1 in our example (Proposition 4).
2. Beyond balls: $h(x)Mx$ is certified for every h exactly when M is P -skew for some $P \in \mathcal{P}(A)$ (Theorem 6); in the plane the Euclidean rotation qualifies iff $A + A^T < 0$ (Proposition 7). Gain can be traded

against local dissipation, with time dependence and ISS (Theorem 10); bursts and switching are covered by Theorems 22 and 24.

3. For normal matrices and whenever $\omega^* = 0$, the quadratic threshold is the exact global threshold $r_{NL} = r_{\mathbb{C}} = r_{\mathbb{R}}$ (Theorem 12). In general $r_{NL} < r_{\mathbb{R}}$ can occur (Proposition 14, Theorem 19); moreover $r_{NL} = r_{TV} \geq r_{\mathbb{C}}$ for $n \geq 3$ (Theorem 17), and $r_{NL} = r_{\mathbb{C}}$ iff $(A, r_{\mathbb{C}})$ is not TV-stable (Theorem 18).

Limitations and open problems

- *The exact global threshold is open when $r_{\mathbb{C}} < r_{\mathbb{R}}$.* For A_3 we only know $0.1671 \leq r_{NL} \leq 0.2001$ (Proposition 20); so a non-quadratic V could gain at most a factor 1.20 over $r_{\mathbb{C}}$. Closing the bracket needs (i) a rigorous non-quadratic certificate for the unstructured sector, e.g. $V = \|x\|e^{w(x/\|x\|)}$ with w a polynomial: the condition is convex in the coefficients, but we have no proof of such a certificate; (ii) a rigorous lower bound for $r_{TV}(A_3)$ matching the upper bound: our optimisation suggests $r_{TV}(A_3) \approx 0.1999$ – 0.2000 (24-knot family: exponent $+9 \cdot 10^{-5}$ at 0.200, $-1.4 \cdot 10^{-3}$ at 0.199), which is not a proof; and (iii) the case $n = 2$ of $r_{NL} = r_{TV}$ (Theorem 17 needs $n \geq 3$). By Theorem 17, for $n \geq 3$ closing the bracket for A_3 is exactly the computation of $r_{TV}(A_3)$, a maximal-Lyapunov-exponent problem for a differential inclusion; Theorem 18 characterizes $r_{NL} = r_{\mathbb{C}}$ by the exponent $\lambda_{\max}(A, r_{\mathbb{C}})$, but we have no practical test for $\lambda_{\max}(A, r_{\mathbb{C}}) = 0$ beyond the sufficient conditions of Theorem 12.
- *Conservatism of the extensions.* The exact-rate Theorem 22 is sharp in Example 23, but this relies on $\omega^* = 0$; for $r_{\mathbb{C}} < r_{\mathbb{R}}$ the gap of Section 6 reappears. Theorem 24 has no numerical benchmark yet.
- *Scope.* Item 1 is classical; the contribution lies in the certified directions, the trade-off theorem and the quantitative comparison of thresholds. State-dependent A and infinite-dimensional settings are not treated.

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