

Bifurcation Analysis and Coexisting Attractors in a Hénon Map with State-Dependent Step Delay

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Abstract

We study a two-parameter family of planar difference equations obtained from the classical Hénon map by letting the delay of the quadratic term depend on the sign of the current state: the term x_n^2 is used when $x_n \geq 0$ and x_{n-1}^2 when $x_n < 0$. The resulting map is discontinuous on the switching line $x_n = 0$ and non-invertible. We prove that the two Hénon fixed points persist but that the saddle one becomes a repeller, that the stable 2-cycle of the Hénon map persists exactly up to $a_* = (1 - b)^2$, and that no 2-cycle exists beyond this value. Numerical experiments for $b = 0.3$ show that, beyond a_* , the attractor is a sequence of stable odd-period cycles whose period decreases as a grows, followed by chaotic regimes, and that several attractors (for instance a 5-cycle and a 10-cycle at $a = 0.9$) coexist with finely interwoven basins of attraction. For $b = 0.3$ we prove, with explicit tail estimates, that stable cycles of every odd period $N \geq 5$ exist on intervals of a that accumulate at a_* geometrically.

Keywords: Hénon map; state-dependent delay; discontinuity-induced bifurcations; multistability; Lyapunov exponent; discrete dynamical systems

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1 Introduction

The Hénon map [1] is a standard example of a two-dimensional system with a strange attractor, and its parameter space has been studied in great detail [2]. Delayed versions of it have also been considered: Sprott [9] and Bilal and Ramaswamy [10] study time-delayed Hénon maps of increasing dimension (the latter report multistability before the onset of chaos in high dimension), El-Sayed et al. [11] treat a Hénon system with two different constant delays, and Caraballo et al. [12] recover Hénon-type complexity in a continuous-time model through delays. In all these works the delay is a fixed constant; the generalized time-delayed Hénon map of [10] is the closest of them to the present one. State-dependent delay maps have also been studied [16], but from a

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different viewpoint (numerical algorithms and dynamics of projections) and not for the planar map considered here. We consider the simplest state-dependent alternative in the Hénon setting: a delay that switches between 0 and 1 according to the sign of the current state. To our knowledge this variant is not treated in the works cited.

Piecewise-smooth systems are usually analysed through border-collision bifurcations [3–5], and non-smooth maps are known to produce sequences of periodic attractors organised by codimension-three bifurcations [18]; for *discontinuous* maps the same term is used and the theory is mostly one-dimensional [13–15], although two-dimensional discontinuous maps with period-adding structures and multistability have also been studied [21]. The map studied here is discontinuous across its switching line (Proposition 1) and two-dimensional, so we speak of discontinuity-induced bifurcations. Multistability is a topic of interest in its own right [6, 7], and coexisting sinks of the Hénon map itself have been located and proved to exist rigorously [19, 20].

Contributions. (i) We give a closed-form analysis of the fixed points and of the 2-cycle (Section 3). (ii) We document numerically, for $b = 0.3$, a cascade of odd-period windows following the loss of the 2-cycle, and coexisting attractors with intertwined basins (Section 4). (iii) We prove this statement for $b = 3/10$ (Theorem 12): for every odd $N \geq 5$ there is an interval of a with an asymptotically stable N -cycle having one negative point, and for $N \geq 49$ the intervals are explicit, disjoint and accumulate at a_* with ratio tending to $1/b$. Statements in Section 3 are proved; the observations in Section 4 are numerical and are labelled as such.

2 The model

Let $a > 0$ and $0 < b < 1$. The *Hénon map with state-dependent step delay* is

$$x_{n+1} = 1 - ax_{n-\tau(x_n)}^2 + y_n, \quad y_{n+1} = bx_n, \quad \tau(x) = \begin{cases} 0, & x \geq 0, \\ 1, & x < 0. \end{cases} \quad (1)$$

Since $y_n = bx_{n-1}$, the state at time n is the pair (x_n, x_{n-1}) , and (1) is the planar map $F(x, z) = (F_1(x, z), x)$ with

$$F_1(x, z) = \begin{cases} 1 - ax^2 + bz, & x \geq 0, \\ f(z), & x < 0, \end{cases} \quad f(u) = 1 + bu - au^2. \quad (2)$$

For $\tau \equiv 0$ one recovers the Hénon map. Write $\Sigma = \{x = 0\}$.

Proposition 1. (a) For $x_n < 0$ one has $x_{n+1} = f(x_{n-1})$, independent of x_n . (b) F is discontinuous at every point $(0, z)$ with $z \neq 0$, with jump $\lim_{x \rightarrow 0^-} F_1 - F_1(0, z) = -az^2$. (c) F is not invertible on $\{x < 0\}$: a point (u, v) with $v < 0$ has up to two preimages, the roots z of $az^2 - bz + u - 1 = 0$.

Proof. (a) is (2). (b) At $x = 0^+$ the first component is $1 + bz$, while for $x \rightarrow 0^-$ it tends to $1 + bz - az^2$. (c) For $v < 0$ a preimage must have $x = v < 0$, hence $u = f(z)$. $\square$

Remark 1. For $x > 0$ the Jacobian determinant is $-b$ (as in the Hénon map); for $x < 0$ the Jacobian is $\begin{pmatrix} 0 & b - 2az \\ 1 & 0 \end{pmatrix}$, with determinant $-(b - 2az)$, which changes sign along $z = b/(2a)$. Both facts contrast with the Hénon map, which is a diffeomorphism [8].

3 Fixed points and the loss of the 2-cycle

Put $D = (1 - b)^2 + 4a$ and

$$a_0 = -\frac{(1-b)^2}{4}, \quad a_1 = \frac{3(1-b)^2}{4}, \quad a_2 = \frac{5-6b+5b^2}{4}, \quad a_* = (1-b)^2.$$

Note $a_1 < a_* < a_2$ for every $b \in (0, 1)$, because $a_* < a_2 \iff 0 < (1+b)^2$.

Proposition 2. For $a > 0$ the map (1) has exactly two fixed points $x_{\pm} = (-(1-b) \pm \sqrt{D})/(2a)$, with $x_+ > 0 > x_-$.

- (i) $P_+ = (x_+, x_+)$ has multipliers given by $\lambda^2 + 2ax_+\lambda - b = 0$; it is asymptotically stable iff $0 < a < a_1$, and a flip bifurcation occurs at $a = a_1$.
- (ii) $P_- = (x_-, x_-)$ has multipliers $\lambda = \pm\sqrt{1 + \sqrt{D}}$, so $|\lambda| > 1$ for all $a > 0$: P_- is a repeller (a saddle in the Hénon map).

Proof. At a fixed point $x = 1 + bx - ax^2$, i.e. $ax^2 + (1-b)x - 1 = 0$, giving $x_{\pm}$. (i) Near P_+ one has $x_n > 0$, so the map coincides with the Hénon map and the Jacobian is $\begin{pmatrix} -2ax_+ & b \\ 1 & 0 \end{pmatrix}$. By the Jury conditions for $\lambda^2 + c_1\lambda + c_0$ with $c_1 = 2ax_+$, $c_0 = -b$: $1 + c_1 + c_0 = \sqrt{D} > 0$; $1 - c_1 + c_0 > 0 \iff \sqrt{D} < 2(1-b) \iff a < a_1$; $|c_0| < 1$. (ii) Near P_- one has $x_n < 0$, so the Jacobian is $\begin{pmatrix} 0 & b - 2ax_- \\ 1 & 0 \end{pmatrix}$ and $\lambda^2 = b - 2ax_- = b + (1-b) + \sqrt{D} = 1 + \sqrt{D} > 1$. $\square$

Proposition 3. Let $0 < b < 1$.

- (i) For $a_1 < a \leq a_*$ the Hénon 2-cycle $\{p, q\}$, with $p + q = (1-b)/a$ and $pq = (1-b)^2/a^2 - 1/a$, has $p, q \geq 0$; it is therefore also a 2-cycle of (1), and it is asymptotically stable.
- (ii) For $a > a_*$ the map (1) has no 2-cycle.

Proof. (i) Imposing $x_{n+1} = x_{n-1} = p$, $x_n = q$ in the Hénon recursion gives $(1-b)p = 1 - aq^2$ and $(1-b)q = 1 - ap^2$. Subtracting gives $a(p+q) = 1-b$ ($p \neq q$); adding gives the stated product. Real roots require $a \geq a_1$. The sum is positive and $pq \geq 0 \iff a \leq a_*$, so $p, q \geq 0$ and the orbit never leaves $\{x \geq 0\}$, where (1) equals the Hénon map. Its multipliers have trace $4(1-b)^2 - 4a + 2b$ and determinant b^2 ; asymptotic stability is equivalent to $a_1 < a < a_2$, and $a_* < a_2$. (ii) Let $\{p, q\}$, $p \neq q$, be a 2-cycle. If $p, q \geq 0$ it is a Hénon 2-cycle, which needs $pq = (1-b)^2/a^2 - 1/a \geq 0$, i.e. $a \leq a_*$; impossible. If $p \geq 0 > q$: from the state $(x_n, x_{n-1}) = (q, p)$ one gets $p = f(p)$, so $1 - ap^2 = (1-b)p$; from (p, q) one gets $q = 1 - ap^2 + bq$, so $(1-b)q = 1 - ap^2 = (1-b)p$ and $q = p$, a contradiction. If $p, q < 0$ both satisfy $u = f(u)$, so both equal x_- . $\square$

Proposition 3 identifies the first discontinuity-induced bifurcation of the system: at $a = a_*$ the point q of the 2-cycle reaches Σ and, for larger a , the cycle cannot continue. For $b = 0.3$ one has $a_1 = 0.3675$, $a_* = 0.49$, $a_2 = 0.9125$.

4 Numerical results

4.1 Methods

All results use $b = 0.3$. Orbits are computed in double precision; an orbit is declared to escape if $|x_n| > 10^8$. Periods are detected on the tail of an orbit after a transient of 3000–60000 iterations with tolerance 10^{-8} – 10^{-10} . Largest Lyapunov exponents λ_1 are obtained by propagating a tangent vector with the Jacobians of Remark 1 ($2 \cdot 10^4$ iterates after a transient of $2 \cdot 10^3$), for the initial condition $(x_0, x_{-1}) = (0.1, 0.1)$; since several attractors may coexist, λ_1 refers to the attractor reached from that initial condition. Basins are computed on regular grids of initial conditions $(x_0, x_{-1}) \in [-1.5, 1.5]^2$.

4.2 Bifurcation diagram and Lyapunov exponent

Figure 1 compares the delayed map with the classical one. For $a \leq a_*$ the two diagrams coincide (period-2 attractor, as predicted by Proposition 3); in the sampled runs ($a = 0.44, \dots, 0.49$, 31×31 grid of initial conditions) a single attractor was found. For $a > a_*$ the diagrams separate: the delayed map shows the odd-period windows described below, whereas the classical map keeps its period-doubling route to chaos. Figure 2 shows

λ_1 for both maps, which is negative on the periodic windows and positive in the chaotic regimes (e.g. $\lambda_1 \approx 0.114$ for the delayed map and -0.281 for the classical one at $a = 0.95$).

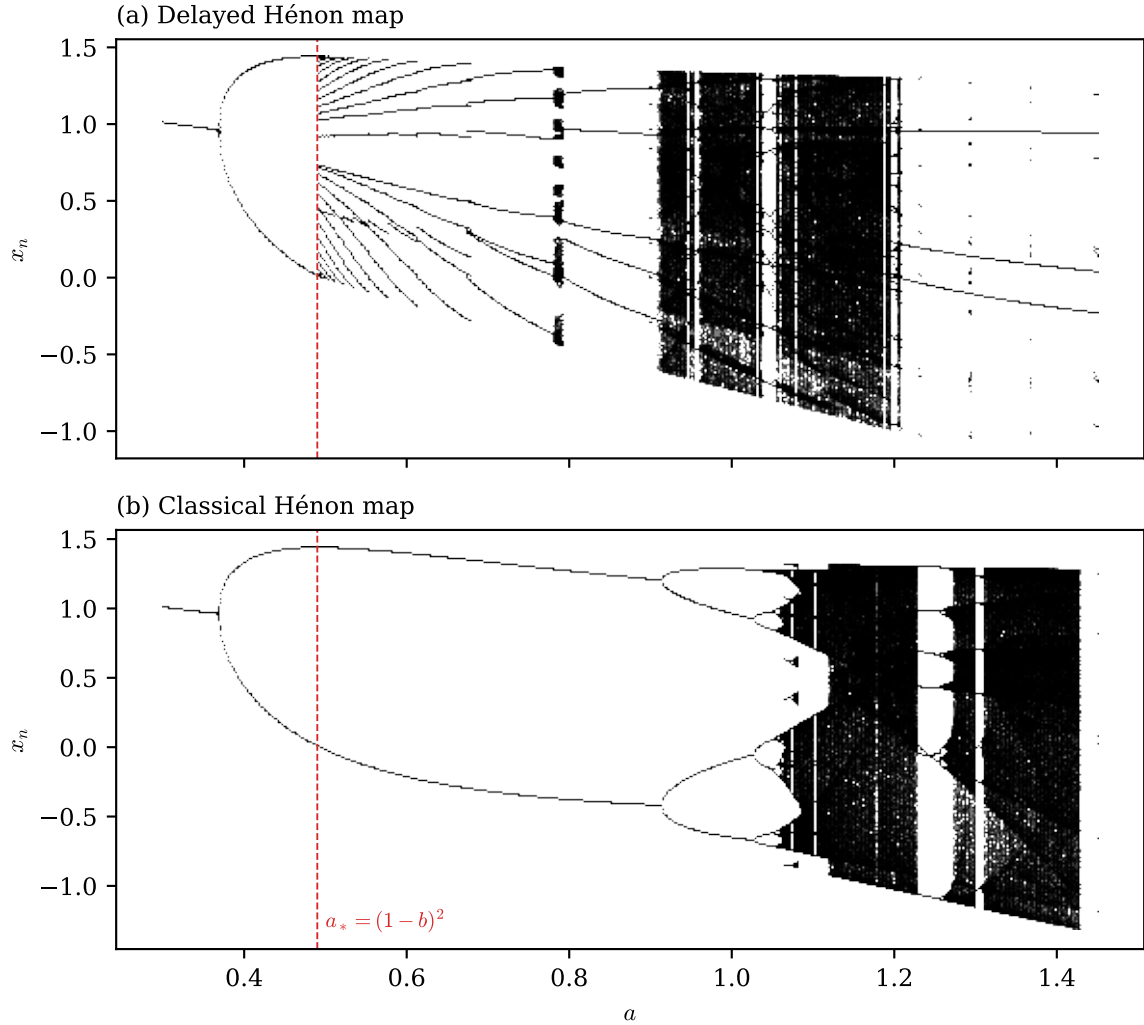


Fig. 1 Bifurcation diagrams for $b = 0.3$ (25 random initial conditions per value of a , last 60 iterates). The dashed line is $a = a_* = (1 - b)^2$.

4.3 Odd-period windows after the loss of the 2-cycle

Just above a_* the attractor is a cycle of high odd period, and the period decreases in steps of two as a grows (Table 1). We found no even-period window and no coexisting attractor in this range with the grid used. Table 1 lists the first sampled value of a (step $5 \cdot 10^{-4}$) at which each period appears for the initial condition $(0.1, 0.1)$; the onset values are therefore accurate only up to the sampling step. For $a \gtrsim 0.78$ a short irregular interval precedes a stable 5-cycle, which appears at $a \approx 0.7905$.

In one-dimensional piecewise-linear discontinuous maps, sequences of periodic windows whose periods change by a fixed step are known as period-adding structures [14, 15]; see [17] for the general theory of period-adding and period-incrementing cascades. The staircase of Table 1, where the odd period decreases in steps of two, is reminiscent of these cascades, but our map is two-dimensional and nonlinear, and we make no claim of equivalence.

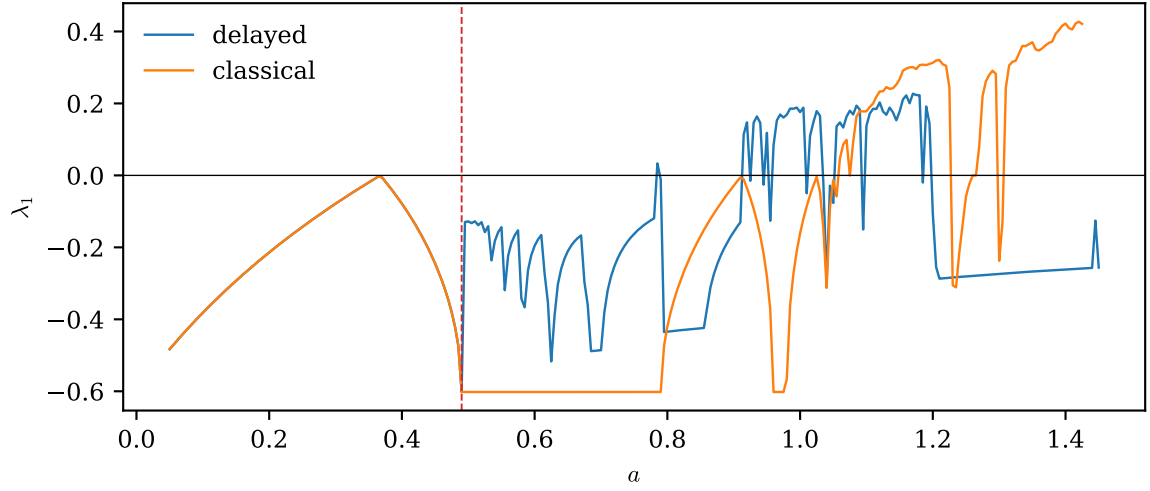


Fig. 2 Largest Lyapunov exponent versus a for $b = 0.3$, initial condition $(0.1, 0.1)$.

period	31	29	27	25	23	21
onset of a	0.4905	0.4915	0.4925	0.4950	0.4985	0.5040
period	19	17	15	13	11	9
onset of a	0.5115	0.5210	0.5335	0.5520	0.5755	0.6120
period	7	5				
onset of a	0.6745	0.7905				

Table 1 Observed odd-period windows of the delayed map ($b = 0.3$), initial condition $(0.1, 0.1)$. A period-35 cycle was also observed at $a = 0.4901$.

Conjecture 4. For $b = 0.3$ the periods of the stable cycles seen for $a \downarrow a_*$ are all odd and grow without bound as $a \downarrow a_*$, so that the windows of Table 1 form an infinite sequence accumulating at a_* .

Lemma 5. Let $N \geq 3$. The map (1) has an N -cycle with exactly one point in $\{x < 0\}$ (and then minimal period N) if and only if there are $u < 0$ and $x_1 > 0$ such that the sequence $x_0 = u$, x_1 , $x_{n+1} = 1 - ax_n^2 + bx_{n-1}$ ($1 \leq n \leq N-1$) satisfies $x_1, \dots, x_{N-1} > 0$ (or ≥ 0) and

$$x_N = u, \quad f(x_{N-1}) = x_1. \quad (3)$$

Proof. Put $x_{-1} = x_{N-1}$. By Proposition 1(a), $x_0 < 0$ gives $x_1 = f(x_{-1}) = f(x_{N-1})$, and for $1 \leq n \leq N-1$ one has $x_n \geq 0$, so the Hénon recursion applies and gives x_N ; the states (x_N, x_{N-1}) and (x_0, x_{-1}) coincide iff $x_N = u$. The converse is the same computation read backwards. A cycle with one negative point per period cannot have a smaller period, since a period $M < N$ dividing N would contain $N/M > 1$ negative points. $\square$

Proposition 6. Let $b = 3/10$. For each pair (N, a) in Table 2, system (3) has a solution (u, x_1) in a box of radius 10^{-12} around the values listed (unique in that box) with $u < 0 < x_1, \dots, x_{N-1}$. Hence (1) has an N -cycle with exactly one point in $\{x < 0\}$, and it is asymptotically stable.

Proof. Existence and local uniqueness: interval Krawczyk test on the box, which maps into its interior. Signs: interval enclosure of $x_0, \dots, x_{N-1}$ over the box. Stability: all points of the cycle lie off Σ , so the map is smooth near them, and the monodromy matrix is the product of the Jacobians of Remark 1 along the cycle; interval

N	a	$u = x_0$	trace	determinant
35	0.4901	-0.00013	-0.0011	$1.83 \cdot 10^{-18}$
31	0.4905	-0.00024	-0.0065	$2.27 \cdot 10^{-16}$
29	0.4915	-0.00150	-0.0135	$2.53 \cdot 10^{-15}$
27	0.4925	-0.00060	-0.0253	$2.81 \cdot 10^{-14}$
25	0.4950	-0.00145	-0.0393	$3.14 \cdot 10^{-13}$
23	0.4985	-0.00097	-0.0519	$3.51 \cdot 10^{-12}$
21	0.5040	-0.00236	-0.0593	$3.94 \cdot 10^{-11}$
19	0.5115	-0.00432	-0.0585	$4.44 \cdot 10^{-10}$
17	0.5210	-0.00387	-0.0448	$5.02 \cdot 10^{-9}$
15	0.5335	-0.00176	-0.0167	$5.68 \cdot 10^{-8}$
13	0.5520	-0.00886	0.0103	$6.51 \cdot 10^{-7}$
11	0.5755	-0.00239	0.0650	$7.46 \cdot 10^{-6}$
9	0.6120	-0.00280	0.1145	$8.68 \cdot 10^{-5}$
7	0.6745	-0.01575	0.1382	$1.04 \cdot 10^{-3}$
5	0.7905	-0.00056	0.1923	$1.27 \cdot 10^{-2}$

Table 2 Verified stable N -cycles with one point in $\{x < 0\}$ ($b = 0.3$); a taken from Table 1 and from the note to it.

enclosures of its trace and determinant satisfy the Jury conditions $|\det| < 1$, $|\text{tr}| < 1 + \det$ over the whole box. $\square$

Proposition 6 settles the existence and stability of the cycles of Table 1 (and of the period-35 cycle) at the tabulated parameter values only; it is a finite computation and says nothing about other N or about the windows in a .

Statement 7. *Prove, for every sufficiently large odd N , the existence and stability of a cycle of period N with one point in $\{x < 0\}$ on an interval of values of a (equivalently, solve (3)), and determine how these intervals accumulate at a_* .*

Theorem 12 below proves this statement for $b = 3/10$: for every odd $N \geq 5$ the cycle exists and is stable on an interval of a , and for $N \geq 49$ these intervals are explicit, disjoint and accumulate geometrically at a_* .

The limiting orbit at a_*

Fix b (all checks are for $b = 0.3$), let $a = a_* + \varepsilon$ with $\varepsilon \in [0, \delta_0]$, and let $p(a) > q(a)$ be the points of the Hénon 2-cycle of Proposition 3. They are analytic near a_* , with $p(a_*) = 1/(1-b)$, $q(a_*) = 0$ and $q'(a_*) = -1/(1-b)^3 < 0$; hence $q(a) < 0$ for $\varepsilon > 0$ (a “virtual” 2-cycle). The monodromy of the 2-cycle has characteristic polynomial $\lambda^2 - T\lambda + b^2$ with $T = 4(1-b)^2 - 4a + 2b$; at a_* the multiplier is b (double), so the 2-cycle is a smooth sink of the Hénon map squared, with a uniform contraction rate $\rho \in (b, 1)$ in a neighbourhood U , for all $\varepsilon \in [0, \delta_0]$.

Let $S(a) = (x_0, x_1) = (q(a), f(p(a)))$ be the state after a visit to the negative point in the limit of infinite period, let $x_n(a)$ be its orbit under the Hénon recursion $x_{n+1} = 1 - ax_n^2 + bx_{n-1}$ (no switching), and put $\sigma_j(a) = x_{2j+1}(a)$, $w_j = \sigma_j - q$.

Proposition 8 (the limiting orbit at a_*). *Let $a = a_* = (1-b)^2$, so $q = 0$, $p = 1/(1-b)$ and $ap = 1-b$, $ap^2 = 1$. Put $u_j = p - x_{2j}$ and $w_j = x_{2j+1}$ for the orbit of $S(a_*) = (0, f(p))$. Then for $j \geq 0$*

$$u_{j+1} = bu_j + aw_j^2, \quad w_{j+1} = bw_j + u_{j+1}(c - au_{j+1}), \quad c = 2(1-b), \quad (4)$$

with $u_0 = p$, $w_0 = bp$. For $b = 3/10$: (a) $0 < u_j < p$ and $w_j > 0$ for all $j \geq 0$, so $x_{2j} > 0$ and $\sigma_j(a_*) > 0$ for $j \geq 1$; (b) $w_{j+1} > bw_j$ for all $j \geq 0$; (c) $u_j, w_j \rightarrow 0$ geometrically, so the orbit tends to the 2-cycle with $x_{2j} \rightarrow p$, $x_{2j+1} \rightarrow q = 0$; (d) $w_{j+1}/w_j \rightarrow b$ from above, hence $w_{j+1} < \theta w_j$ for $j \geq j_0$ and any $\theta \in (b, 1)$.

Proof. For $n \geq 1$ the orbit follows the Hénon recursion. Writing $x_{2j} = p - u_j$, and using $p = 1 + bp$ and $1 - ap^2 = 0$ (both hold at a_*), gives $u_{j+1} = bu_j + aw_j^2$; likewise $x_{2j+3} = 1 - ap^2 + 2apu_{j+1} - au_{j+1}^2 + bw_j$ gives the second relation in (4), with $2ap = c$. Exact rational arithmetic gives $(u_1, w_1) = (0.51857\dots, 0.72280\dots)$ and $(u_5, w_5) = (0.27775\dots, 0.53263\dots)$, and confirms (4) against the direct orbit for $j \leq 5$.

Boxes. Let $B_1 = \{0 \leq u \leq 0.42, 0 \leq w \leq 0.55\}$. If $(u, w) \in B_1$ then $u' \leq b(0.42) + a(0.55)^2 < 0.275 \leq 0.42$, and since $u \mapsto u(c - au)$ is increasing for $u \leq c/(2a)$, $w' \leq b(0.55) + 0.275(c - 0.275a) < 0.513 \leq 0.55$. So B_1 is invariant; and $(u_5, w_5) \in B_1$. For $1 \leq j \leq 5$ the values are positive and $u_j < p$ by the exact computation (they lie in the analogous invariant box $\{u \leq 0.52, w \leq 0.73\}$, which one checks the same way).

(a) $u_{j+1} = bu_j + aw_j^2 > 0$ and, as long as $u_{j+1} \leq p$, $w_{j+1} = bw_j + u_{j+1}(c - au_{j+1}) \geq bw_j + (1 - b)u_{j+1} > 0$ because $c - ap = 1 - b$. Since $u_j \leq 0.52 < p$ in the boxes, the claims follow by induction.

(b) $w_{j+1} - bw_j = u_{j+1}(c - au_{j+1}) > 0$.

(c) In B_1 , with $\lambda = 0.7$ and $L = w + \lambda u$, using $w' \leq bw + cu'$: $L' \leq bw + (c + \lambda)(bu + aw^2) = [b + (c + \lambda)aw]w + (c + \lambda)bu \leq 0.9L$, because $b + 2.1 \cdot a \cdot 0.55 < 0.867$ and $(c + \lambda)b = 0.63 = 0.9\lambda$. So $L_j \leq C0.9^j$.

(d) Let $r_j = u_j/w_j$ and $c_0 = c - 0.42a > 0$. From (c), $w_j \rightarrow 0$. By (4), $w_{j+1} \geq bw_j + c_0u_{j+1}$ and $u_{j+1}/w_j = br_j + aw_j$, so $r_{j+1} \leq \varphi(br_j + aw_j)$ with $\varphi(s) = s/(b + c_0s)$, which satisfies $\varphi' \leq 1/b$ and $\varphi(br) = r/(1 + c_0r)$. Hence $r_{j+1} \leq r_j/(1 + c_0r_j) + (a/b)w_j$. Since $\sum w_j < \infty$, r_j is bounded, and taking $\limsup$ gives $\rho \leq \rho/(1 + c_0\rho)$, i.e. $r_j \rightarrow 0$. Finally $w_{j+1}/w_j = b + u_{j+1}(c - au_{j+1})/w_j \leq b + c(br_j + aw_j) \rightarrow b$. $\square$

Proposition 8 describes the limit orbit at $\varepsilon = 0$: $w_j \sim (A + Bj)b^j$ with $B > 0$ (Jordan block at the double multiplier b). For $\varepsilon > 0$ the argument must be redone, which is the content of the next subsection.

Proof of the Statement for $b = 3/10$

Throughout this subsection $b = 3/10$, $a_* = 49/100$, $a = a_* + \varepsilon$ with $\varepsilon \in E = [0, \delta_0]$, $\delta_0 = 1.01 \cdot 10^{-7}$, and $p > 0 > q$ are the points of the Hénon 2-cycle (Section 3), $c = 2ap$. From $pq = -\varepsilon/a^2$ we get the exact formula $|q| = Q(\varepsilon) := \varepsilon/(a^2p)$, which is increasing on E and satisfies $Q(\varepsilon) = 2.9154\dots \varepsilon(1 + O(\varepsilon))$.

Lemma 9 (exact reduced recursion). *Let $x_{2j} = p - u_j$, $x_{2j+1} = q + w_j$. The Hénon recursion $x_{n+1} = 1 - ax_n^2 + bx_{n-1}$ is equivalent, for every ε and every initial pair, to $v_{j+1} = \Phi(v_j)$, $v_j = (u_j, w_j)$,*

$$u' = bu + 2aqw + aw^2, \quad w' = bw + cu' - au'^2.$$

Its linear part at 0 is $M = \begin{pmatrix} b & 2aq \\ cb & b + 2acq \end{pmatrix}$, with $\det M = b^2$ and trace $2b\tau$, $\tau = 1 - acQ/b \in (-1, 1]$; write $\Phi(v) = Mv + R(v)$.

Proof. Use $p = 1 - aq^2 + bp$, $q = 1 - ap^2 + bq$ and expand; this is an exact polynomial identity (checked also symbolically/numerically). $\square$

Lemma 10 (cycle equations). *Let $K \geq 1$, $N = 2K + 1$ and let $\Psi_K = \Phi^K$. A cycle as in Lemma 5 corresponds to a fixed point of*

$$\Theta(v_0) = (p - q - w_K, b(p - q) + (c - b)u_K - au_K^2), \quad (u_K, w_K) = \Psi_K(v_0),$$

via $x_0 = p - u_0$, $x_1 = q + w_0$, plus the sign conditions $x_{2j} > 0$ ($1 \leq j \leq K$), $x_{2j+1} > 0$ ($0 \leq j \leq K - 1$) and $x_N = q + w_K < 0$. The monodromy matrix is $DT = SD\Psi_K \text{diag}(-1, f'(z))$, $z = x_{N-1}$, with S a signed permutation.

Proof. The conditions $x_N = x_0$ and $f(x_{N-1}) = x_1$ of Lemma 5 read $u_0 = p - q - w_K$ and, using $f(p - u) = f(p) + (c - b)u - au^2$, $f(p) - q = b(p - q)$, $w_0 = b(p - q) + (c - b)u_K - au_K^2$. The return map is $T(x_0, x_{-1}) = (q + w_K, p - u_K)$ with $v_0 = (p - x_0, f(x_{-1}) - q)$. $\square$

Lemma 11 (tail estimates). *Let $\kappa = (\|M\|_\infty + b)/b$, $S = (1 + b)/(1 - b)^3$, and suppose $|R(v)| \leq C_R|v|^2$, $|R(v) - R(v')| \leq C'_R \max(|v|, |v'|)|v - v'|$ for $|v|, |v'| \leq \tilde{d} := 2\kappa d$, and $v = 4\kappa^2 C_R d S/b \leq 1$, $v' = 4\kappa^2 C'_R d S/b \leq 1$. Then for two orbits with $|v_J|, |v'_J| \leq d$ (same ε) and $m = n - J \geq 0$: (i) $|v_n| \leq 2\kappa d(m + 1)b^m$; (ii) $|v_n - v'_n| \leq 2\kappa(m + 1)b^m|v_J - v'_J|$; (iii) $|v_n - M^m v_J| \leq 4\kappa^3 C_R S d^2 m b^{m-1}$.*

Proof. By Cayley–Hamilton, $M^k = b^{k-1}(U_{k-1}(\tau)M - bU_{k-2}(\tau)I)$ with Chebyshev polynomials of the second kind, $|U_j| \leq j+1$ on $[-1, 1]$, hence $\|M^k\| \leq \kappa(k+1)b^k$. Write $v_n = M^n v_J + \sum_{k=J}^{n-1} M^{n-1-k} R(v_k)$ and use induction on n : inserting $|v_k| \leq 2\kappa d(k-J+1)b^{k-J}$ gives $\sum \leq 4\kappa^3 C_R S d^2 m b^{m-1}$, which is $\leq \kappa d(m+1)b^m$ iff (roughly) $v \leq 1$; this proves (i) and (iii). The same argument applied to $v_n - v'_n$, with $R(v_k) - R(v'_k)$, gives (ii) under $v' \leq 1$. $\square$

Theorem 12 ($b = 3/10$). Put $J = 22$, $K_0 = 24$, $\zeta = 0.0122$, and let $u_{\pm}, w_{\pm}$ be the enclosures of u_J, w_J in Table 3. Define $W_n^- = (1 - \zeta)b^{n-J}(w_- + (n-J)cu_-)$ and $W_n^+ = (1 + \zeta)b^{n-J}(w_+ + (n-J)cu_+)$.

- (a) For every odd $N \geq 5$ there is a nonempty open interval of values $a > a_*$ for which (1) has an asymptotically stable cycle of period N with exactly one point in $\{x < 0\}$.
- (b) For $K \geq K_0$ and $N = 2K + 1$ one can take $I_K = \{a_* + \varepsilon : W_K^+ < Q(\varepsilon) < W_{K-1}^-\}$. These intervals are nonempty, pairwise disjoint, ordered (I_{K+1} lies below I_K) and accumulate at a_* ; the ratio of consecutive lower endpoints tends to $1/b$, and $\varepsilon \approx (\beta(K - n_0))b^K/|q'(a_*)|$ up to a factor 1 ± 0.015 , with $\beta = 1.842 \cdot 10^4$, $n_0 = 9.998$, $|q'(a_*)| = (1 - b)^{-3} = 2.9154 \dots$

Interval-arithmetic certificate. The certificate (interval arithmetic with outward rounding) does the following. (A) It encloses v_j , $0 \leq j \leq J$, and $D_{v_0} v_J$ over the box $\{\varepsilon \in E, |v_0 - v^*(\varepsilon)|_{\infty} \leq r\}$, $v^* = (p - q, b(p - q))$ (the limit orbit), and checks $p - u_j > 0$ ($1 \leq j \leq J$) and $w_j > Q(\delta_0)$ ($0 \leq j \leq J$). (B) It computes κ, C_R, C'_R, v, v' with $d \geq \sup |v_J|$. (C) It checks the inequalities used below. The results are in Table 3; all checks return True.

δ_0	$1.01 \cdot 10^{-7}$	r	$4.21 \cdot 10^{-6}$
$d \geq \sup v_J $	$6.95 \cdot 10^{-7}$	$\Lambda = \sup \ D_{v_0} v_J\ _{\infty}$	$2.15 \cdot 10^{-5}$
$u_J \in$	$[4.1221, 4.1353] \cdot 10^{-8}$	$w_J \in$	$[6.9277, 6.9481] \cdot 10^{-7}$
$\kappa \leq$	3.4001	S	3.7901
$C_R \leq, C'_R \leq$	0.7302, 1.4603	$v \leq, v' \leq$	$3.0 \cdot 10^{-4}, 6.0 \cdot 10^{-4}$
$L_{K_0} = 2\kappa\Lambda(K_0 - J + 1)b^{K_0 - J} \leq$	$3.95 \cdot 10^{-5}$	$A_{K_0} = 2\kappa d(K_0 - J + 1)b^{K_0 - J} \leq$	$1.28 \cdot 10^{-6}$
$\zeta_{\rho} = 4\kappa^3 C_R S d^2 / (bcu_-) \leq$	0.01214	$\sup_K \zeta \leq$	0.01214
window ratio $W_K^+ / W_{K-1}^- \leq$	0.3341	$Q(\delta_0) \geq W_{K_0-1}$	True

Table 3 Constants of the certificate ($J = 22$, $K_0 = 24$).

Proof. [Proof of Theorem 12(b)] Fix $K \geq K_0$, $\varepsilon \in E$ and put $L_K = \Lambda \cdot 2\kappa(K - J + 1)b^{K-J}$, which is decreasing in K , so $L_K \leq L_{K_0}$.

Step 1 (existence). For v_0, v'_0 in the ball $B(v^*, r)$ (inside the box), $|v_J|, |v'_J| \leq d$ and $|v_J - v'_J| \leq \Lambda|v_0 - v'_0|$, so Lemma 11(ii) gives $\text{Lip}(\Psi_K) \leq L_K$. With $\text{mult} = \max(1, c - b + 2aA_K) \leq 1.11$ one has $\text{Lip}(\Theta) \leq \text{mult}L_K \leq 4.4 \cdot 10^{-5} < 1/2$, and $|\Theta(v^*) - v^*| \leq \text{mult}A_K \leq 1.42 \cdot 10^{-6} \leq r/2$, where $A_K \geq |v_K(v^*)|$ by Lemma 11(i). Hence Θ maps $B(v^*, r)$ into itself and is a contraction: it has a unique fixed point v_0^c .

Step 2 (stability). By Lemma 10, $\|DT\|_{\infty} \leq \text{mult}L_K < 1$, so both multipliers lie in the open unit disc.

Step 3 (signs). v_0^c lies in the box, so v_j^c for $j \leq J$ is enclosed in step (A): $x_{2j} > 0$, $x_{2j+1} > 0$ for $j \leq J$. For $j > J$, $|u_j| \leq 2\kappa d < p$ by Lemma 11(i), so $x_{2j} > 0$. For the odd points, for $n = J + m \leq K$ write $(M^n v)_2 = b^m(w + mcu + e)$. Using $U_{m-1}(\cos \theta) = \sum_{j=0}^{m-1} \cos((m-1-2j)\theta)$ one gets $0 \leq m - U_{m-1} \leq \theta^2 m(m^2 - 1)/6$, and since $1 - \tau = acQ/b$, $\theta^2 \leq 2s(1 + s)$, $s = acQ/b$,

$$|e| \leq \frac{\theta^2 m^3}{6}(cu + w) + \frac{2ac}{b}Qmw.$$

In the window $Q \leq \hat{q}_K := 2b^{K-1-J}(w_+ + (K-1-J)cu_+)$, so $|e|/(w_- + mcu_-) \leq \zeta_e(K)$, which is non-increasing for $K \geq J + 4$ and negligible: with $m = K - J$, $\hat{q}_{K+1}/\hat{q}_K \leq bm/(m-1)$, so the two terms of the numerator of ζ_e are multiplied by at most $b(m+1)^3/(m^2(m-1))$ and $b(m+1)/(m-1)$, both ≤ 1 for $m \geq 4$ (the first because $7m^3 - 19m^2 - 9m - 3 \geq 0$ for $m \geq 4$), while the denominator increases; $\sup_K \zeta_e$ is then the maximum

over $K_0 \leq K \leq J+3$ and the next values, computed in the certificate. Here $(M^m v)_2 = b^m [cU_{m-1}u + (U_{m-1} - U_{m-2})w - \frac{2acQ}{b}U_{m-1}w]$ with $U_j = U_j(\cos \theta)$, $0 \leq m - U_{m-1} \leq \theta^2 m(m^2 - 1)/6$, and $0 \leq 1 - (U_{m-1} - U_{m-2}) = 1 - \frac{\cos((m-1/2)\theta)}{\cos(\theta/2)} \leq \frac{\theta^2(m^2-m)}{2\cos(\theta/2)} \leq \frac{\theta^2 m^3}{6}$ (equality of the first form is $\sin m\theta - \sin(m-1)\theta = 2\cos((m-\frac{1}{2})\theta)\sin(\frac{\theta}{2})$), which gives the bound on $|e|$ above. Together with Lemma 11(iii) ($\zeta_\rho \leq 0.01214$), $w_n^c \in [W_n^-, W_n^+]$ for $J < n \leq K$. Since $b(w + (m+1)cu) < w + mcu$ (because $w_-/u_+ > 0.6$), W_n^- is decreasing in n . If $W_K^+ < Q < W_{K-1}^-$, then $w_n^c > Q$ for $J < n \leq K-1$ and $w_K^c < Q$, i.e. $x_{2n+1} > 0$ ($n \leq K-1$) and $x_N = q + w_K^c < 0$. By Lemma 10 the fixed point is the required cycle.

Step 4 (windows). By the median inequality, $W_K^+/W_{K-1}^- \leq b \max(\frac{w_+ + cu_+}{w_-}, \frac{u_+}{u_-}) \frac{1+\zeta}{1-\zeta} \leq 0.3341 < 1$, so the set $\{W_K^+ < Q < W_{K-1}^-\}$ is a nonempty open interval; it lies in E because $Q(\delta_0) \geq W_{K_0-1} \geq W_{K-1}^-$. Disjointness and ordering follow from $W_K^- \leq W_K^+$. The ratio $W_K^+/W_{K+1}^+ \rightarrow 1/b$ because the factor $1 + \zeta$ cancels and $(w + mcu)/(w + (m+1)cu) \rightarrow 1$. Finally $b^{K-J}(w_J + (K-J)cu_J) = \beta b^K(K - n_0)$ with β, n_0 obtained from the $\varepsilon = 0$ orbit (Proposition 8); the error band is ζ plus the $\leq 10^{-3}$ spread of the enclosures. $\square$

Proof. [Proof of Theorem 12(a)] For $N \geq 2K_0 + 1 = 49$ this is (b). For odd $5 \leq N \leq 47$ it is the finite computation of Proposition 6: Table 2 covers $N \in \{5, \dots, 31, 35\}$, and Table 4 covers $N = 33$ and $37 \leq N \leq 47$ (same interval Krawczyk test, with the starting guess taken from the limit orbit). In all cases the cycle is asymptotically stable with one negative point, so these properties persist on an open interval of a . $\square$

N	a	$u = x_0$
33	0.490259682758	$-3.04 \cdot 10^{-4}$
37	0.490033170396	$-4.01 \cdot 10^{-5}$
39	0.490011340807	$-1.39 \cdot 10^{-5}$
41	0.490003811637	$-4.70 \cdot 10^{-6}$
43	0.490001265189	$-1.57 \cdot 10^{-6}$
45	0.490000415908	$-5.19 \cdot 10^{-7}$
47	0.490000135656	$-1.70 \cdot 10^{-7}$

Table 4 Additional cycles verified as in Proposition 6; the Krawczyk test, the signs and the Jury conditions all hold.

Remark 2 (on the earlier uniformity hypothesis). For fixed $\varepsilon > 0$ the multipliers of the 2-cycle are *complex* with modulus b ($\det = b^2$, $\tau < 1$), so w_j eventually changes sign (numerically at $j = 38$ for $\varepsilon = 10^{-3}$ and at $j = 18$ for $\varepsilon = 10^{-2}$). A hypothesis of the form “ $w_j > w_{j+1} > 0$ for all $j \geq j_0$, uniformly in a ” is therefore false; what is needed, and what is proved above, is control for $j \leq K$ with $\varepsilon \lesssim \varepsilon_{K-1}$, where the rotation angle $\theta \sim \sqrt{\varepsilon}$ satisfies $K\theta \rightarrow 0$.

j	ε_j	$\varepsilon_{j-1}/\varepsilon_j$	$1/b$
15	$4.25 \cdot 10^{-4}$	2.49	3.33
20	$2.20 \cdot 10^{-6}$	3.00	3.33
25	$8.03 \cdot 10^{-9}$	3.11	3.33
30	$2.60 \cdot 10^{-11}$	3.17	3.33

Table 5 Roots ε_j of $\sigma_j(a_* + \varepsilon)$ ($b = 0.3$, 60 digits, ordinary multiprecision). The ratio approaches $1/b$ from below, as expected from $(A + Bj)b^j$. The roots are ordered, $\varepsilon_j < \varepsilon_{j-1}$, for $4 \leq j \leq 31$, and σ'_j at the root tends to $q'(a_*) = -2.9155$ ($-5.44, -2.918, -2.9155$ for $j = 10, 20, 30$).

Remark 3 (discrepancy for small N , quantified). Table 6 lists windows of the *limit* orbit. The true lower border of the N -cycle (computed by Newton’s method in 60 digits, $x_0 = 0^-$) lies below ε_K by 0.11% ($N = 35$), 0.67% (31), 1.4% (29), 2.5% (27), 3.7% (25), 4.8% (23), 5.8% (21) and 6.6% (19). The cause is the sensitivity of σ_K

$N = 2K + 1$	predicted window $(\varepsilon_K, \varepsilon_{K-1})$	sampled onset $a - a_*$	inside?
35	$(5.64 \cdot 10^{-5}, 1.59 \cdot 10^{-4})$	$1.0 \cdot 10^{-4}$	yes
31	$(4.25 \cdot 10^{-4}, 1.06 \cdot 10^{-3})$	$5.0 \cdot 10^{-4}$	yes
29	$(1.06 \cdot 10^{-3}, 2.39 \cdot 10^{-3})$	$1.5 \cdot 10^{-3}$	yes
27	$(2.39 \cdot 10^{-3}, 4.82 \cdot 10^{-3})$	$2.5 \cdot 10^{-3}$	yes
25	$(4.82 \cdot 10^{-3}, 8.71 \cdot 10^{-3})$	$5.0 \cdot 10^{-3}$	yes
23	$(8.71 \cdot 10^{-3}, 1.44 \cdot 10^{-2})$	$8.5 \cdot 10^{-3}$	no
21	$(1.44 \cdot 10^{-2}, 2.22 \cdot 10^{-2})$	$1.40 \cdot 10^{-2}$	no
19	$(2.22 \cdot 10^{-2}, 3.28 \cdot 10^{-2})$	$2.15 \cdot 10^{-2}$	no

Table 6 Predicted windows against the sampled onsets of Table 1.

to the initial state: $\partial \sigma_K / \partial x_0 = -1.5 \cdot 10^{-3}, -5.9 \cdot 10^{-5}, -2.1 \cdot 10^{-7}$ for $K = 17, 20, 25$, i.e. of order $10^5 Kb^K$, the large constant coming from the expanding transient. For $N \geq 49$ this shift is below 10^{-6} and is absorbed in the factor ζ of Theorem 12; the exact borders for $N = 49, \dots, 81$ were checked to lie inside the predicted bands.

4.4 Coexisting attractors

At the values sampled with step 0.01 in $0.44 \leq a \leq 0.89$ only one attractor was found on the initial-condition grids, but at other values several periodic attractors coexist. All cases of Table 7 were confirmed with long runs ($6 \cdot 10^4$ iterations, tolerance 10^{-9} on the tail). In the classical map at $a = 0.9$ the same sampling gives a single attractor (the 2-cycle), so the multistability is a consequence of the delay mechanism. Figure 3 shows the basins: the sets of initial conditions leading to different attractors are finely interwoven over the whole sampled square.

a	attractors	share of initial conditions	escape
0.90	P5, P10	83.9%, 16.1%	0%
1.23	P3, P9	77.9%, 20.0%	2.1%
1.45	P3, P7	50.4%, 38.8%	10.8%

Table 7 Coexisting periodic attractors ($b = 0.3$), computed on a 360×360 grid of $(x_0, x_{-1}) \in [-1.5, 1.5]^2$. At $a = 1.29$ a 3-cycle and a 12-cycle also coexist (found on a 31×31 grid).

4.5 Chaotic attractor

For $a = 1.15$ both maps have a positive exponent (0.182 for the delayed map, 0.269 for the classical one). Figure 4 shows that the geometry of the delayed attractor differs visibly from the classical one. For a in $[1.21, 1.45]$ (sampled with step 0.01) a stable 3-cycle was observed, and the delayed map still has bounded attractors at $a = 1.45$, where the classical orbit from $(0.1, 0.1)$ escapes.

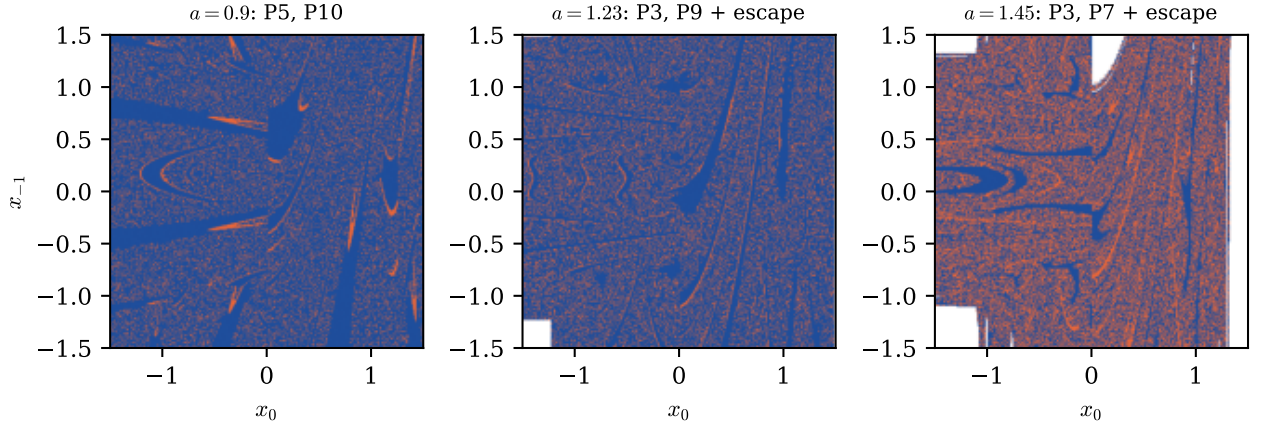


Fig. 3 Basins of attraction in the (x_0, x_{-1}) plane ($b = 0.3$). Colours distinguish the attractors named in the panel titles; white is escape to infinity.

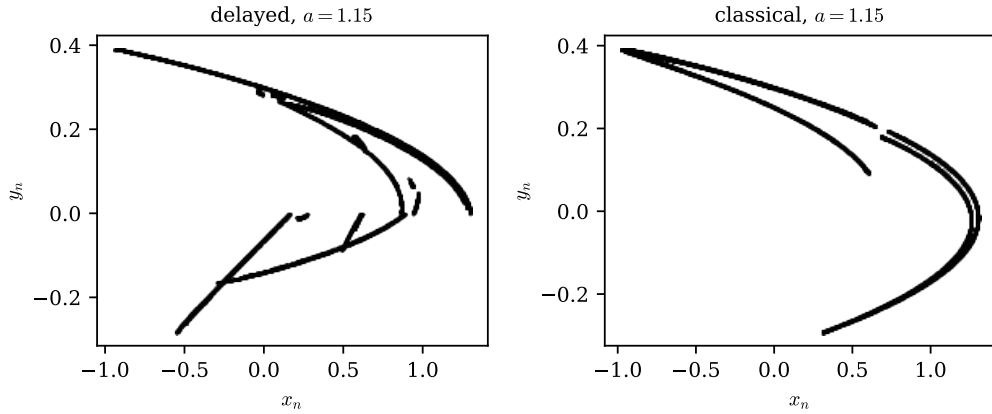


Fig. 4 Attractors in the (x_n, y_n) plane for $a = 1.15$, $b = 0.3$, initial condition $(0.1, 0.1)$.

5 Conclusions

A single bit of state dependence in the delay, namely whether the quadratic term sees the present or the previous value, is enough to change the Hénon map qualitatively. The map becomes discontinuous and non-invertible, its saddle turns into a repeller, and the period-two attractor of the Hénon map survives only until its second point reaches the switching line, which happens exactly at $a_* = (1 - b)^2$. Beyond a_* the picture changes for $b = 0.3$. The attractor becomes a cycle of high odd period, and the period decreases in steps of two as a grows; chaotic bands follow, and then several periodic attractors coexist with finely interwoven basins, something not seen in the classical map at the values we sampled. The staircase is not only an observation: for $b = 3/10$ every odd period $N \geq 5$ is realised on an interval of a , and for $N \geq 49$ these windows are explicit, disjoint and shrink towards a_* with ratio $1/b$, the contraction rate of the Hénon 2-cycle at a_* . All results are for $b = 0.3$, and the proof is specific to $b = 3/10$. The theorem shows that the odd-period cycles exist and are stable, but not that they are the only attractors for those values of a , nor that even-period windows are absent.

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