

Singularization Problems for Projective Clusters in High-Dimensional Algebraic Geometry and Their Applications to Complex Domains

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Abstract. The singularization solution of high-dimensional algebraic geometric projective clusters is based on minimal models. In this paper, we obtain a singularization theorem of projective clusters from the Kähler metric, combined with the singularization theorem of projective clusters of Siu-Yau and Mok under the different conditions that the Ricci curvature of a 2-dimensional complete noncompact Kähler projective cluster manifold is positive and bounded when M is a n -dimensional complete noncompact Kähler projective cluster manifold. M is biholomorphic pure with a proposed projective cluster if and only if $k_r(x_0) \geq -\frac{c}{1+r^2}$ is satisfied and the general Sobolev inequalities $\|f\|_i \leq C_0 \|\nabla f\|_q^\theta \|f\|_s^{1-\theta} \quad \forall f \in C_0^\infty(M)$, $\frac{1}{t} = \frac{\theta}{p} + \frac{1-\theta}{s}$, and $\int_M \text{Ric}^n < \infty$ are satisfied. Then we introduce L^2 all-pure multiple inverse canonical cross section estimates, Bezout estimates and Gauss-Bonnet integral estimates to prove the validity of the singularization theorem. And combining the singularization theorem with the related theorem of high-dimensional algebraic clusters, we design the singular branching solution problem of rooted functions, in order to illustrate the feasibility of the application of the singularization of projective clusters to the complex domain.

1. Introduction

Algebraic geometry has been at the core of modern mathematics. It is not only crucial to the development of the discipline of mathematics, but also widely used in modern physics, engineering technology, electronic computers and other fields [1-2]. In algebraic geometry, an algebraic cluster represents the set of common zeros of a set of polynomials, and algebraic clusters are the most fundamental objects of study in algebraic geometry, and when an algebraic cluster can be defined by exactly one equation, we call it a hyperquadratic surface [3-4]. Hyperquadratic surfaces are the most common and important class of algebraic clusters, and their simple and beautiful geometric structures not only occupy an important position in the study of algebraic clusters, but also have important applications in production practice [5-6]. Classification of algebraic clusters is a central problem in algebraic geometry. Classification of high-dimensional algebraic

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clusters has been a hot issue in the study of algebraic geometry in recent decades, and it mainly applies the theory of minimal models established by Japanese scholar Mori, which is the core of the theory of birational equivalence classification of high-dimensional algebraic clusters, and it is an effective tool for the study of high-dimensional algebraic clusters [7-8].

Throughout the history of algebraic geometry, the focus of research has been swinging between algebra and geometry. Literature [9] combines abstract mathematical concepts with real-world problems in a lecture format, guiding the audience to expand their horizons, cross boundaries, and explore with an open and inclusive mind the many topics that are closely related to, but go beyond, the finite projective space. Literature [10] suggests that the Euclidean PGA provides a natural transition between vector space modeling and more complex and powerful CGAs in scientific research and teaching, which not only meets a variety of practical needs, but also has the prospect of a wide range of applications. Literature [11] proposed novel mathematical formulas and related methods that can be applied to the field of inorganic compounds by using network theoretical methods and standard conformal geometric algebra, which is of great significance for further development of mathematical chemistry and theoretical chemistry, and provides an effective tool for understanding the connection between the properties and structures of inorganic compounds. Literature [12] discusses the problems encountered in the process of coupled clustering calculations through the algebraic geometry method, and gives relevant solutions and examples to verify them. The study further enriches and improves the knowledge system about the application of coupled set theory in high-precision computational quantum chemistry methods, and lays the foundation for the in-depth exploration of the relevant fields in the future.

Based on combing the relevant definitions of high-dimensional algebraic clusters, the article introduces the Kähler metric to define the singularization problem of projective clusters on complex domains, and proves the singularization theorem by combining the L^2 all-pure many times inverse canonical cross section estimation, the Bezout estimation and the Gauss-Bonnet integral estimation inference. In addition, regular algebraic clusters and smooth projective clusters of high-dimensional algebraic geometry are derived. Based on the singularization theorem, the singular branch of a rooted function is solved, and a logical argument is made for the application of the singularization theorem for projective clusters to complex domains.

2. Basic Theory

2.1. Definition of high-dimensional algebraic clusters

Algebraic geometry focuses on the study of the geometric properties of algebraic clusters and the classification of algebraic clusters. The study of algebraic curves and surfaces has a long history and is very rich in theory and results. However, the systematic study of high-dimensional algebraic clusters is only in the last two decades or so, and as Clemens points out, the study of three-dimensional algebraic clusters was generally regarded as hopelessly complex and incomprehensible until the pioneering work of Shigehito Mori [13].

Let X be a smooth algebraic cluster, and if the canonical cluster K_M of X is not digitally effective, then by Mori theory there are extremal rays R on X and a contractive morphism $\phi : M \rightarrow X$ that contracts the curve of all digits in M belonging to R equivalently to a point in X . If K_M is digitally valid, then there is no contraction state map, and the dual rational state map from X to any algebraic cluster is isomorphic, so X is “minimal”, and X is said to be minimal. The minimal model scheme constructs a minimal model in the birational equivalence class of every non-directed cluster, so that every algebraic cluster in the equivalence class can be contracted to a minimal model. In turn, every algebraic cluster in the equivalence class can be obtained by a finite number of enlargements of the minimal model. The key to studying the problem of very small models is to study the contraction morphisms $\phi : M \rightarrow X$, i.e., to study the structure of the set of exceptions $E = \{x \in M \mid \phi \text{ not isomorphic at } x \text{ points}\}$ of ϕ . When $\dim M = 3$, Mori makes a complete classification of the structure of E , thus proving that the minimal model conjecture holds in three dimensions.

There are special algebraic clusters that have been studied very systematically in higher dimensions, whose geometrical structures are the simplest of the higher dimensional clusters and are fully known, and

these special algebraic clusters are the cornerstones for the study of more complex algebraic clusters in general. The following are some common special algebraic clusters:

1. P^n denotes a n -dimensional projective space.
2. Q^n denotes a hyperquadratic surface in P^{n+1} .
3. Let M be a regular cluster in n dimensions and L a rich line cluster on M . If $-K_M$ rich, call M a Fano cluster, if $K_M \approx -(n-1)L$, call (M, L) a DelPezzo cluster, and if $K_M \approx -(n-2)L$, call (M, L) a Mukai cluster.

Suppose that there exist full state projections $\phi : M \rightarrow X$, whose fibers are all connected, and X is a regular cluster in dimension m . If there exists a rich line cluster H on X such that $K_M + (n-m+1)L$ is linearly equivalent to the pullback of H , i.e., $K_M + (n-m+1)L \approx \phi^*(H)$, then (M, L) is said to be a vortex on X .

If $K_M + (n-m)L \approx \phi^*(H)$, then (M, L) is said to be a hyperquadratic surface fibration on X .

If $K_M + (n-m-1)L \approx \phi^*(H)$, then (M, L) is said to be a Pezzo fibration on X .

If $K_M + (n-m-2)L \approx \phi^*(H)$, then (M, L) is said to be a Mukai fibration on X .

2.2. Preparatory knowledge of related theorems

This paper is always discussed on complex domains [14].

Let X be a n -dimensional ($n \geq 3$) smooth projective cluster and K_X a canonical cluster of X . We define:

$$N_1(X) = \{1\text{-cycles} / \approx\} \otimes \sim \quad (1)$$

$$N^1(X) = \{X \text{ on the divider } \approx\} \otimes \sim \quad (2)$$

$NE(X)$ is the closure of the convex cone generated by the effective 1-cycles in $N_1(X)$.

$NE_{K_X} = \{Z \in NE(X) \mid K_X \cdot Z \geq 0\}$, i.e., the negative part of $NE(X)$.

$\rho(X) = \dim N^1(X)$ is called the Picard number of X .

where $\sim$ is the real number field and $\approx$ denotes the numerical equivalence relation.

Definition 2.1. Let D be a divisor on X . Call D numerically valid if the number of intersections of each curve C with D is nonnegative, i.e., $D \cdot C \geq 0$; call D rich if and only if for any $Z \in NE(X) - \{0\}$, there is $D \cdot Z > 0$.

Definition 2.2. The semiline $R = \mathbb{R}_+[C]$ in $NE(X)$ is called an extremal semiline, where C is a curve on X if $K_X \cdot C < 0$, and $C_1, C_2 \in R$, follow from $C_1 + C_2 \in R$ for any $C_1, C_2 \in NE(X) - \{0\}$.

Definition 2.3. A projective manifold X such that $-K_X = c_1(X)$ is a rich clump of lines is called a Fano manifold. Given a Fano manifold X , the largest integer that divides $c_1(X)$ in $H^2(X, \mathbb{Z})$ is called the Fano indicator of X . If the Fano metric of a Fano manifold X is equal to $\dim X - 1$, the Fano manifold X is said to be a DelPezzo manifold.

Theorem 2.4. Let X be a n -dimensional smooth projective cluster and $NE(X)$ be a minimal closed convex cone containing NE_{NX} and all extremal half-lines. For any open convex cone U containing $NE_{Sx} - \{0\}$, there are only finitely many extremal half-lines not contained in U . Each extremal semiline R can be generated by a rational curve C , i.e. $R = \mathbb{R}_+[C]$, and $1 \leq -K_X \cdot C \leq \dim X + 1$, such that C is called an extremal rational curve. As many extreme half-lines $R_i (i \in I)$ can be chosen such that

$$NE(X) = NE_{K_X} + \sum_{i \in I} R_i.$$

Theorem 2.5. Let X be a n -dimensional smooth projective cluster and R an extremal semilinear, then there exists $f : X \rightarrow Y$, f a fiber of full state projections to some regular projective cluster Y which is connected such that $-K_X f^*$ is rich. And if C is any irreducible curve on X , then $f(C)$ is a point when and only when $[C] \in R$, and f is said to be a contraction state shot accompanied by an extremal semiline R , denoted f_R . Further, there is $\rho(Y) = \rho(X) - 1$. Setting $f_R : X \rightarrow Y$ such that $E = \{x \in X \mid f_R \text{ is not an isomorphism at the point } x\}$, then E is a concatenation set of all curves in X contracted to a point by f_R , and is called the set of exceptions of f_R . When $\dim E = n$ this is the case, it is called a fiber-type contraction, when $\dim E = n - 1$ f_R and f_R are doubly rational morphisms, it is called

a divisor-type contraction, and when $\dim E \leq n - 2$ f_R and f_R are doubly rational morphisms, it is called a minor contraction. The length of an extremal semiline is defined as:

$$l(R) = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } X \text{ and } [C] \in R\} \quad (3)$$

By the cone theorem, we know that $0 < l(R) \leq n + 1$.

Theorem 2.6. Let X be a n -dimensional smooth projective fiber and R an extremal semiline on X . Let $f_R : X \rightarrow Y$ be a contraction state map with extremal semiline R , E be an exception set of a contraction state map f_R , $f_{R|E} : E \rightarrow f_R(E)$ be its restriction state map, and F be an irreducible branch of any one of the fibers of $f_{R|E}$. Then we have

$$\dim E + \dim F \geq \dim X + l(R) - 1.$$

Theorem 2.7. Let X be a smooth n -dimensional Fano cluster, i.e., $-K_X$ be rich, and the overall length of X be defined as $l(X) = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } X\}$. $X \cong P^n$ if and only if $l(X) \geq n + 1$ holds for any $C \subset X$.

Theorem 2.8. Let X be a n -dimensional light-absorbing shadow cluster and L be a rich line cluster, then there are:

1. If $-K_X \approx (n + 1)L$, then $(X, L) \cong (P^n, \mathcal{O}_{P^n}(1))$.
2. If $-K_X \approx nL$, then $(X, L) \cong (Q^n, \mathcal{O}_{Q^n}(1))$.

Theorem 2.9. Let $\varepsilon = \mathcal{O}_{P^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{P^1}(a_r)$ be a vector clump of rank r on P^1 , then:

$$c_1(\varepsilon) \cdot C \geq r = \text{rank}(\varepsilon) \quad (4)$$

The integer $a_1 \geq \cdots \geq a_r$ is uniquely determined by ε . For any rational curve C in X .

Theorem 2.10. Let $L \in \text{Pic}(X)$ be digitally valid but $K_X + L$ not digitally valid, then there exist extremal semilines C such that $(K_X + L) \cdot C < 0$ and $-K_X \cdot C \leq n + 1$.

2.3. The singularization problem for projective clusters

In higher dimensional algebraic geometry projective clusters, if the deflection of a metric under the Hermite liaison is equal to 0 means that this metric is a Kähler metric. In complex Finsler geometry, the deflection under the Chern-Finsler liaison has a horizontal part and a mixed part, which leads to the definition of the strong Kähler metric, the Kähler metric, and the weak Kähler metric, respectively, based on the fact that different parts of the deflection under the Chern-Finsler liaison are equal to zero [15].

Existing studies proved that the Kähler metric and the strong Kähler metric are equivalent, focused on the You-invariant complex Finsler metric, and computed the equations satisfied by the You-invariant complex Finsler metric that is a weak Kähler metric, but did not find a non-Kähler weak Kähler metric. Thus, the question about the existence of non-Kähler weak Kähler metrics in complex Finsler geometry remains open.

Let F be a complex Finsler metric and $G = F^2$ the corresponding fundamental tensor. Let $\Gamma_\eta^\gamma = G^{3\gamma} \Gamma_\mu^\gamma v^\mu$, $G_{\gamma\bar{\mu}}$, and $\Gamma_{v,\mu}^\gamma$ be obtained by computing the horizontal Chern-Finsler liaison coefficients and the flat Berwald liaison coefficients denoted as $\Gamma_{v,\mu}^\gamma$ and $G_{v\mu}^\gamma$, respectively, that is:

$$G_{\gamma\mu} = \frac{\partial G^\gamma}{\partial v^\mu}, \quad \Gamma_{v,\mu}^\gamma = \frac{\partial \Gamma_{i\mu}^\gamma}{\partial v^v}, \quad G_{v\mu}^\gamma = \frac{\partial G_{\gamma\mu}}{\partial v^v} \quad (5)$$

F is said to be a complex Berwald metric if $\Gamma_{v,\mu}^\gamma$ depends only on the bottom manifold coordinate $z = (z^1, \dots, z^n)$, and F is said to be a weakly complex Berwald metric if $G_{v\mu}^\gamma$ depends only on the bottom manifold coordinate $z = (z^1, \dots, z^n)$.

Theorem 2.11. Let $F = r\phi(t, s)$ be a You-invariant strongly pseudoconvex complex Finsler metric on region $D \subset \mathbb{C}^n$. Then F is a weakly complex Berwald metric if and only if:

$$\phi_t + [1 - tg(t)s + g(t)s^2]\phi_s = [f(t) + g(t)s]\phi \quad (6)$$

where $f(t)$ and $g(t)$ are two real-valued smooth functions. The solution of the above equation satisfying the initial condition $\phi(0, s) = \varphi(s)$ is:

$$\phi(t, s) = \exp\left(\int_0^t f(\tau) d\tau\right) \varphi\left(\frac{s-t}{h(t, s)}\right) h(t, s) \quad (7)$$

where $\varphi(s)$ is a given real-valued smooth function and $h(t, s)$ has the expression:

$$h(t, s) = \exp\left(\int_0^t \tau g(\tau) d\tau\right) + (s-t) \int_0^t g(\sigma) \exp\left(\int_0^\sigma \tau g(\tau) d\tau\right) d\sigma \quad (8)$$

Theorem 2.12. Let $F = r\phi(t, s)$ be a weakly complex Berwald metric on the You-invariant region $D \subset \mathbb{C}^n$. Remembering that the total pure curvature under this metric is K_F , we have:

1. $K_F = 0$ if and only if:

$$\phi(t, s) = \varphi(s-t) \quad (9)$$

where $\varphi(w)$ is a positive smooth function satisfying Theorem 8.

2. $K_F = \text{constant} \neq 0$ if and only if:

$$\phi(t, s) = \left(\frac{B}{B-At}\right)^2 [A(s-t) + B] \quad (10)$$

where A, B are two real-valued constants and satisfy $A \neq 0, B > 0, B-At > 0$. Under this metric there is then:

$$K_F = -4AB^{-2} \quad (11)$$

Theorem 2.13. Let $F = r\phi(t, s)$ be a quotient-invariant strongly pseudoconvex complex Finsler metric on region $D \subset \mathbb{C}^n$. Then F is a Kähler-Finsler metric if and only if:

$$\begin{cases} \phi_{ss} = 0 \\ \phi_s - \phi_t + s\phi_{st} = 0 \end{cases} \quad (12)$$

The solution of this equation is easily obtained as:

$$\phi(t, s) = a_0(t) + a'_0(t)s \quad (13)$$

where $a_0(t)$ is a positive smooth function and satisfies $a_0(t) + ta'_0(t) > 0$.

Theorem 2.14. Let $F = r\phi(t, s)$ be a hundred-invariant strongly pseudoconvex complex Finsler metric on region $D \subset \mathbb{C}^n$, then F is a weak Kähler metric if and only if:

$$\begin{aligned} &(\phi - s\phi_s)[\phi + (t-s)\phi_s][\phi_s - \phi_t + s(\phi_{st} + \phi_{ss})] \\ &+ s(t-s)\phi_{ss}[\phi(\phi_s - \phi_t) + s\phi_s(\phi_t + \phi_s)] = 0 \end{aligned} \quad (14)$$

Theorem 2.15. Let M be a complex manifold of projective clusters on a single connective and $F = \alpha + |\beta|$ a complex Berwald Randers metric on M . If F has fanwise totally pure curvature, then M must be one of the following manifolds:

1. $M = \mathbb{CP}^n$, at which point F is the Fubini-Study metric.
2. $M = \{z \in \mathbb{C}^n : |z| < 1\}$, at which point F is a Bergman metric.
3. $M = \mathbb{C}^n$, at which point F is a Minkowski metric.

2.4. Singularization theorem for projective clusters

Siu-Yau initially studied the singularization problem on complete Kähler projective cluster manifolds with finite volume and truncated curvature between two negatives, and later Mok systematically studied the singularization problem on complete Kähler manifolds.

Theorem 2.16. *Let M be a n -dimensional complete noncompact Kähler projective cluster manifold with positive Ricci curvature, and constants C_1, x_0, C_2 there:*

$$R(x) \leq \frac{C_1}{d^2(x_0, x)} \quad (15)$$

$$\int_M \text{Ric}^n < \infty \quad (16)$$

$$\text{vol}(B(x_0, r)) \geq C_2 r^{2n} \quad (17)$$

Here $d(x_0, x)$ denotes the distance from x_0 to x , vol denotes the volume of the measured Earth B , Ric and R denote the Ricci curvature form and the quantitative curvature of M , and then M is doubly holomorphic pure with a proposed projective cluster.

Theorem 2.17. *Let M be a n -dimensional complete noncompact Kähler projective cluster manifold with positive Ricci curvature and a finite topological type, and there exist a constant k, k', k'' and a real number p such that for any $r > 0$, then:*

$$R(x) \leq \frac{k}{d^p(x_0, x)} \quad (18)$$

$$\int_M \text{Ric}^n < \infty \quad (19)$$

$$\int_B \omega^n (1 + d(x_0, x))^{np} \leq k' \log r \quad (20)$$

$$|\text{Bisect}(v, w)| \leq k'' \text{Ric}(v, \bar{v}) \quad (21)$$

Here ω and $\text{Bisect}(\cdot, \cdot)$ denote Ricci forms and holomorphic bisecting curvature with unit tangent vectors $v, w \in T_x^{1,0}(M)$, $x \in M$. Then M is bis holomorphic with a proposed projective cluster.

In this paper, we obtain a singularization theorem under different conditions as follows:

Theorem 2.18. *Let M be a n -dimensional complete noncompact Kähler projective cluster manifold whose Ricci curvature is positively bounded and satisfies the following conditions:*

1. $k_r(x_0) \geq -\frac{c}{1+r^2}$, $k_r(x_0)$ denote the truncated curvature of the geodesic with x_0 as center and r as radius.
2. Satisfies the general Sobolev inequalities $\|f\|_t \leq C_0 \|\nabla f\|_q^\theta \|f\|_s^{1-\theta} \forall f \in C_0^\infty(M)$, $\frac{1}{t} = \frac{\theta}{p} + \frac{1-\theta}{s}$.
3. $\int_M \text{Ric}^n < \infty$.

Then M is doubly holomorphic with a proposed projective cluster.

3. Key Findings

3.1. Proof of the singularization theorem

The singularization problem and the singularization theorem for high-dimensional algebraic geometric projective clusters were given in Section 2.4, and this chapter proves this singularization theorem based on the L^2 full pure multiple inverse canonical section estimate, the Bezout estimate, and the Gauss-Bonnet

integral estimate, which provide support for the realization of the solution of the singularization problem [16-17].

The space $\bigcup_{q>0} \Gamma^2(M, L^{-q})$ forms a subalgebra and gives the local coordinate system of any point on M and separates the points on M . By the case of $k = 1$ in the theorem and by Fatou's lemma:

$$\int_{S=q} \text{Ric}^{n-1} \leq \liminf_{\delta \rightarrow 0} \int_M S_e(S) \wedge \text{Ric}^{n-1} \leq cq \int_M \frac{\text{Ric}^n}{(n-1)!} < +\infty \quad (22)$$

That is, if:

$$\text{vol}([S = 0] \cdot \text{Ric}^{n-1}) \leq \frac{q}{2\pi} \int_M \text{Ric}^n \quad (23)$$

Then $\dim \Gamma^2(M, L^{-q}) \leq cq^n$, so by (22):

$$\dim \Gamma^2(M, L^{-q}) \leq cq^n \quad (24)$$

$R(M, L^{-1})$ constitutes a subdomain of the subpure function domain.

Order:

$$R(M, L^{-1}) = \left\{ \frac{s}{t} \mid s, t \in \Gamma^2(M, L^{-q}), q \in \mathbb{Z}_+, t \neq 0 \right\} \quad (25)$$

Let $S_0, \dots, S_n$ be a holomorphic section in $\Gamma^2(M, L^{-q})$ ($q > 0$ sufficiently large) such that $\left\{ \frac{S_1}{S_0}, \dots, \frac{S_n}{S_0} \right\}$ is a locally holomorphic coordinate system of the x_0 point, and $R(M, L^{-1})$ is a finite dilation field on $C\left(\frac{S_1}{S_0}, \dots, \frac{S_n}{S_0}\right)$ by (25) and the Poincare-Siegel method, by the eigenelement theorem:

$$R(M, L^{-1}) = C\left(\frac{S_1}{S_0}, \dots, \frac{S_n}{S_0}, g\right) \quad (26)$$

where g is a subpure function in $R(M, L^{-1})$ and is algebraic with respect to $C\left(\frac{S_1}{S_0}, \dots, \frac{S_n}{S_0}\right)$. If we take the same denominator, we can assume $g = \frac{S_{n+1}}{S_0}$, when $S_0, \dots, S_{n+1} \in \Gamma^2(M, L^{-q})$ ($q > 0$ is sufficiently large). Now consider the mapping:

$$F : M \rightarrow P^{n+1}, \quad F(x) = [S_0 : \dots : S_{n+1}], \quad x \in M \quad (27)$$

Since g is algebraic, the minimal polynomial of g is:

$$g^p + \sum_{j=0}^{p-1} R_j\left(\frac{S_1}{S_0}, \dots, \frac{S_n}{S_0}\right) g^j = 0 \quad (28)$$

where R_j is a rational function of n variables. Substituting $g = \frac{S_{n+1}}{S_0}$ into the above equation and approximating the denominator yields the chi-square $P(S_0, \dots, S_{n+1}) = 0$ with respect to $S_0, \dots, S_{n+1}$. Let Z_0 be the hypersurface defined by $P(S_0, \dots, S_{n+1}) = 0$ in P^{n+1} , Z be the connected branch containing $F(M \setminus Q)$ in Z_0 , and here Q be the indeterminate set of F .

3.2. Main results and proof

1. Main Theorem I

Let M be an n -dimensional regular algebraic cluster with terminal singularities, L be a rich line cluster of M , and K be a canonical divisor or canonical stratum on M . Suppose K_M is not of nef. Then the effective value r of (M, L) is a real number, which we define to be $\tau = \min\{r \in \mathbb{R}, K + rL\}$, and $\sigma : M \rightarrow W$ is a nef-valued morphism determined by (M, L) . If $\tau = \frac{u}{v}$, $u = n - 1$ are given, then there are: $A = (v - 1)K_M + (k - 1)L$, and:

- a) (M, A) is a DelPezzo cluster, $L \cong vA$.
- b) $(M, A)\sigma : M \rightarrow W$ is a quadratic fiber map on a smooth curve and all fibers are of equal dimension.
- c) $(M, A)\sigma : M \rightarrow W$ is a scroll surface on a regular surface.
- d) $(M, A)\sigma : M \rightarrow W$, is a P^{n-2} -line cluster on a regular surface, and F is a fibered compact map on an extremal ray.
- e) $\sigma : M \rightarrow W$ is a compact map of a disjoint smooth point on a disjoint divisor $E_i \cong P^*$ such that $E_i \subset \text{Reg}(X)_{\mathfrak{D}_E}(E_i) \cong \mathfrak{D}_p \cdot (-1)$, and $A_i \cong \mathfrak{D}_p.(1)$ ($i = 1, 2 \cdots t$). In more detail, $A_n := (\sigma A)$, and $K_\pi + uA_n$ are enriched, and $K_x + uA \approx \sigma^*(K_\pi + uA_n)$.

2. Proof of the main theorem I

Since Assumption $\tau = \frac{u}{v}$, is a nef-value of (M, L) , and by the theorem already Assumption $v \geq 1, u = n-1$, then there is $K_M + (n-1)L$ which is nef and big. Then there is σ which is doubly rational, then it contracts some extreme value curve. Then by the theorem given before, there exists an extremal curve C such that $(K_M + (n-1)L) \cdot C = 0$, taking $p : M_R \rightarrow Y$ to be a contraction map over the extremal curve R , then there is $\sigma = \alpha \circ p$. The relation is shown in Fig. 1.

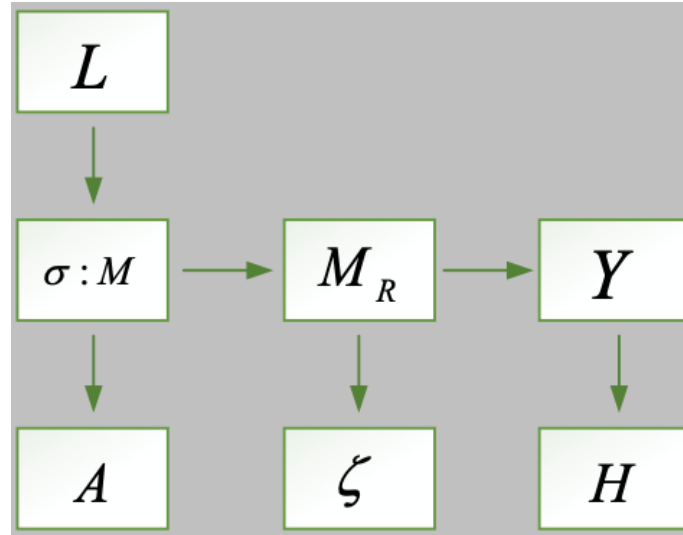


Figure 1: σ diagram

Then by the theorem it follows that $K_M + (n-1)A \approx p^*\zeta$, $\zeta = \alpha^*H$, H are rich clusters of lines over L , and there is $K_M + (n-1)A \approx \sigma^*H$.

From the known condition that $\tau = \frac{u}{v}$ is a nef-value of (M, L) , there exists p, q such that $vp - vq = 1$, $A = qK_M + pL$, it follows that $vA = v(qK_M + pL) = q\sigma^*H - uqL + pvL = q\sigma^*H + L$, and A is abundant, then there is $K_M + uA = K_M + u(qK_M + pL) = pvK_M + upL = p\sigma^*H$.

Speaking further:

$$A_W := (\sigma, A) \quad (29)$$

Let E be the trace of R to obtain $-\sigma$, $K_M + (n-1)A \approx \sigma^*(K_M + (n-1)L)$, $KME \approx -(n-1)AE \approx \mathfrak{D}_{PS^{-1}}(1-n)$, $EE \approx \mathfrak{D}_E(-1)$.

Since $K_M + (n-1)A$ is mundane on E , that gives $(K_M + E)_E \approx K_E \approx \mathfrak{D}_E(-n)$, which in turn gives $E_i \cong P^{n-1}$, $E_i \subset \text{reg}(M)$.

We find that M contains a finite number of divisors such that $\mathfrak{D}_{E_i}(E_i) \cong \mathfrak{D}_{P^{-1}}(-1)$, $AE_1 \cong \mathfrak{D}_{PR^1}(1)$, ($i = 1, \dots, t$), E_i , can be contracted to a smooth point p_i .

Now take $A_i \cong \mathfrak{D}_p.(1)$. Since A is enriched, σ is an isomorphism other than a finite set of points, and L is enriched, then there is $K_M + (n-1)A \approx \sigma^*H$.

The proof is complete.

3. Main Theorem II

Let X be a n -dimensional smooth projective cluster and E be a rich vector bundle of rank $n - 2$ such that $K_X + c_1(E)$ is not numerically valid and $\Lambda(E, K_X) \geq 2$, then (X, E) is one of the following cases:

$$(X, E) \cong (P^n, \oplus^{n-2} \mathcal{O}_{P^n}(1)) \quad (30)$$

$$(X, E) \cong (P^n, \oplus^{n-3} \mathcal{O}_{P^n}(1) \oplus \mathcal{O}_{P^n}(2)) \quad (31)$$

$$(X, E) \cong (Q^n, \oplus^{n-2} \mathcal{O}_{Q^n}(1)) \quad (32)$$

4. Proof of Main Theorem II

Let X be a n -dimensional smooth projective cluster and E be a rich vector clump of rank $n - 2$ such that $K_X + c_1(E)$ is not numerically valid and $\Lambda(E, K_X) \geq 2$, then (X, E) cannot be the following case. That is, (X, E) is a scroll on a smooth curve W , X is a linear P^{n-1} -cluster on W , $g : X \rightarrow W$, and for every fiber F of g , there is $(F, E|_F) \cong (P^{n-1}, \oplus^{n-1} \mathcal{O}_{P^{n-1}}(1))$.

Suppose (X, E) is a scroll on a smooth curve W , X is a linear P^{n-1} -clump on W , $g : X \rightarrow W$, and for every fiber F of g , there is $(F, E|_F) \cong (P^{n-1}, \oplus^{n-1} \mathcal{O}_{P^{n-1}}(1))$. Then let R be an extremal semilinear companion to g , and by the contraction theorem, the Picard number of X is $\rho(X) = \rho(W) + 2 = 4 \wedge (E, K_X) \geq 4$. Because of $-K_X$, is enriched, and there exists another contracted morphism $f : X \rightarrow Z$. Let R' be an extremal semilinear companion to f , and let $R' = \mathbb{R}_+[C]$ and $l(R') = -K_X \cdot C$, then:

$$l(R') = -K_X \cdot C = c_1(E) + 2 \geq \text{rank}(E) + 2 = n \quad (33)$$

Let A be the set of exceptions to f , then:

$$2 \dim A \geq \dim X + l(R') - 1 \geq 2n - 1 \quad (34)$$

And so:

$$\dim A = \dim X = n \quad (35)$$

Since X is irreducible, $A = X$, i.e., $f : X \rightarrow Z$, is a fiber-type contraction. Let F' be a fiber of f , then $\dim F' \geq \dim X + l(R') - \dim A = l(R') - 2 \geq n - 2$. If $\dim F' = n$, then Z is a point, and so $\rho(X) = 2$, a contradiction. Hence $\dim F' = n - 2$, and hence Z is a smooth curve.

Let F be a fiber of $g : X \rightarrow W$. Then by Assumption $F \cong P^{n-1}$. If $f(F) = 0$, then there exists in F a rational curve C' contracted to a point by both g and f , whereupon the numerical equivalence class of C' is $[C'] \in R \cap R' = \phi$, a contradiction. Hence $\dim f(F) \geq 2$, which gives the restriction $f|_F$ on F is a contraction state projective of F to $F : F \rightarrow Z'$. Then by the contraction theorem we have the Picard number $\rho(F) = \rho(Z') + 2 = 4$ of F .

But on the other hand, $F \cong P^{n-1}$, hence $\rho(F) = \rho(P^{n-1}) = 2$, contradiction.

The proof is complete.

3.3. Univariate problem solving

After determining the two main theorems for projective clusters of high-dimensional algebraic geometry, the singularization problem for projective clusters is solved in conjunction with the main theorem of singularization. Based on the singularization theorem together with the main theorems I and II, the following singular value branches of the commonly used projective clusters of rooted functions of type $R(z)$ are obtained:

$$R(z) = R(z_0)^{1/2} \exp \left\{ i \frac{\arg R(z_0)}{2} \right\} \cdot \left| \frac{R(z)}{R(z_0)} \right|^{1/2} \exp \left\{ i \frac{\Delta_L \arg R(z)}{2} \right\} \quad (36)$$

Consider below the case when α is a complex number. Let $\alpha = \alpha_1 + i\alpha_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ (the set of real numbers).

In this case there is:

$$\begin{aligned}
 [R(z)]^\alpha &= [R(z)]^{\alpha_1 + i\alpha_2} = [R(z)]^{\alpha_1} \cdot [R(z)]^{i\alpha_2} \\
 &= [R(z)]^{\alpha_1} \cdot \exp\{i\alpha_2 \text{Ln}[R(z)]\} \\
 &= [R(z)]^{\alpha_1} \cdot \exp\{i\alpha_2 [\ln |R(z)| + i(\arg R(z) + 2k\pi)]\} \\
 &= [R(z)]^{\alpha_1} \cdot \exp \left\{ i\alpha_2 \left[\ln |R(z)| + i \left(\sum_{j=1}^n m_j \arg(z - a_j) - \sum_{k=1}^n n_k \arg(z - b_k) \right) + 2k\pi \right] \right\}
 \end{aligned} \tag{37}$$

By the pivot decision theorem for logarithmic function $\text{Ln } Q(z)$, and combining it with the related theorem on projective clusters of higher dimensional algebraic geometry, we know that a_j, b_k are both its pivots, and ∞ is its pivot when $\alpha_1 \sum_{j=1}^n m_j - \sum_{k=1}^n n_k \neq z$ (integer) or $\alpha_2 \sum_{j=1}^n m_j - \sum_{k=1}^n n_k \neq 0$.

For a multi-valued function of the form $[R(z)]^{\alpha_1 + i\alpha_2}$, the following conclusion follows:

1. a_j ($j = 1, 2, \dots, u$), b_k ($k = 1, 2, \dots, v$) are its fulcrums;
2. None of the finite points except a_j, b_k is its fulcrum;
3. A sufficient condition for an infinite point to be its fulcrum is $\alpha_1 \sum_{j=1}^n m_j - \sum_{k=1}^n n_k \neq z$ (integer) or $\alpha_2 \sum_{j=1}^n m_j - \sum_{k=1}^n n_k \neq 0$. Based on the above theorem, the expression for the single-valued branch L^2B in this case can be obtained as:

$$[R(z)]^{\alpha_1 + i\alpha_2} = [R(z_0)]^{\alpha_1 + i\alpha_2} \cdot \left(\frac{R(z)}{R(z_0)} \right)^{\alpha_1} \exp \left\{ i\alpha_2 \ln \frac{R(z)}{R(z_0)} + \Delta_L \arg R \right\} \tag{38}$$

EXAMPLE: Determine the branch point of the rooted function $F(z) = z^2 - 2$ in the higher dimensional algebraic geometric projective cluster and find the value of the single-valued analytic branch at $z = i$ when z_0 takes $1/2$.

Solution: $F(z)$ can be written as $(z^2 - 2)^{1/2}$, where $m_1 = m_2 = \sqrt{2}, a_1 = -\sqrt{2}, a_2 = \sqrt{2}$, since $\alpha m_1 + \alpha m_2 = 1$ is a constant, ∞ is not its branch point. The branch can be taken to cut through the complex plane C as shown in Fig. 2.

The single-valued branching region G is obtained by:

$$[R(z)]^\alpha = |R(z_0)|^\alpha \exp\{i\alpha(\arg R(z_0))\} \cdot \left| \frac{R(z)}{R(z_0)} \right|^\alpha \exp\{i\alpha\Delta_L \arg R(z)\} \tag{39}$$

Can be obtained as $R(z) = z^2 - \sqrt{2}$, then the single-valued branch satisfying the initial condition is:

$$F_{K_0}(z) = |R(\sqrt{2})|^{1/2} \exp \left\{ i \frac{\arg R(\sqrt{2})}{2} \right\} \cdot \left| \frac{R(z)}{R(\sqrt{2})} \right|^{1/2} \cdot \exp \left\{ i \frac{\Delta_L \arg R(z)}{2} \right\} \tag{40}$$

Since at $z = i$, then:

$$\begin{aligned}
 F_{K_0}(i) &= |R(\sqrt{2})|^{1/2} \exp \left\{ i \frac{\arg 3}{2} \right\} \cdot \left| \frac{R(i)}{3} \right|^{1/2} \cdot \exp \left\{ i \frac{\Delta_L \arg(z - \sqrt{2}) + \Delta_L \arg(z + \sqrt{2})}{2} \right\} \\
 &= |R(i)|^{1/2} \exp \left\{ i \frac{\Delta_L \arg(z - \sqrt{2}) + \Delta_L \arg(z + \sqrt{2})}{2} \right\}
 \end{aligned} \tag{41}$$

where there is a continuous variation of z along L from point $z = 3$ to $z = i$:

$$\begin{cases} \Delta_L \arg(z - \sqrt{2}) = \frac{3\pi}{5} \\ \Delta_L \arg(z + \sqrt{2}) = \frac{2\pi}{5} \end{cases} \tag{42}$$

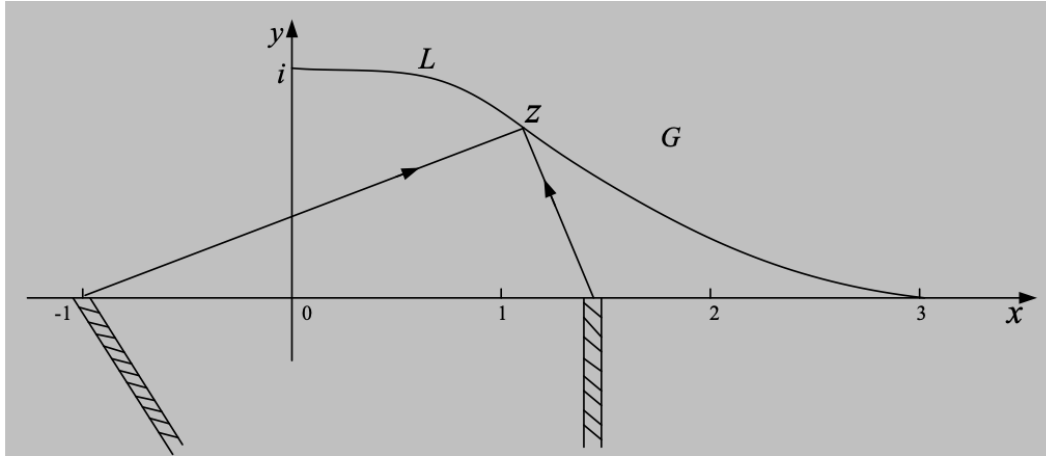


Figure 2: Single-value branch field by taking fulcrum-cutting line

Therefore:

$$F_{K_0}(i) = |R(i)| \cdot \exp \left\{ i \frac{3\pi/5 + 2\pi/5}{2} \right\} = 2\sqrt{2} \exp \left\{ i \cdot \frac{\pi}{2} \right\} = 2\sqrt{2}i \quad (43)$$

In conclusion, it is feasible to utilize the singularization theorem combined with the related theorem of projective clusters of high-dimensional algebraic geometry for the solution of radical functions, which provides a certain research basis for expanding the application scope of high-dimensional algebraic geometry.

4. Conclusion

In this paper, we study the singularization of high-dimensional algebraic geometric projective clusters in different manifolds based on the Kähler metric and explore the specific case of applying it to complex domains.

1. Assuming that M is a n -dimensional complete noncompact Kähler projective cluster manifold with positive bounded Ricci curvature and satisfying condition $k_r(x_0) \geq -\frac{c}{1+r^2}$ and general Sobolev inequalities $\|f\|_q \leq C_0 \|\nabla f\|_q^\theta \|f\|_s^{1-\theta} \forall f \in C_0^\infty(M)$, $\frac{1}{t} = \frac{\theta}{p} + \frac{1-\theta}{s}$, and $\int_M \text{Ric}^n < \infty$, then M is biholomorphic to a proposed projective cluster.
2. Based on the singularization theorem, combine the singularization problem of projective clusters and transform it into the problem of solving the singular-valued branch of a rooted function. For a multi-valued function of the form $[R(z)]^{a_1+ia_2}$, it is known that a_j ($j = 1, 2, \dots, u$), b_k ($k = 1, 2, \dots, v$) are its single-valued branch points, and the finite points except a_j, b_k are not its single-valued branch points, and the sufficient condition for an infinite point to be its branch point is $\alpha_1 \sum_{j=1}^n m_j - \sum_{k=1}^n n_k \neq z$ (integer) or $\alpha_2 \sum_{j=1}^n m_j - \sum_{k=1}^n n_k \neq 0$.

In this paper, although a different theorem is obtained in solving the singularization problem of projective clusters in higher dimensional algebraic geometry, and it is applied to the singular solution of radical functions. However, relatively only one case of application to the complex domain is considered, and the study will be enriched by further extension of the relevant theorems and introduction of algebraic geometry systems under more generalized and special conditions in the subsequent studies.

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