

Symbolic Computation of Exact Solutions of Nonlinear Mathematical Physics Equations Based on a Geometric Grid Factorization Algorithm

Bin Chen^a, Xiaohan Yuan^{b,*}

^aInstitute of Applied Mathematics, Hebei Academy of Sciences, Shijiazhuang, Hebei, 050000, China

^bNanjing Agricultural University, Nanjing, Jiangsu, 210095, China

Abstract. Exploring exact solutions of non-linear mathematical physics equations can help solve problems in more engineering fields. In this paper, the exact solutions of the periodic exact traveling wave solution of the rod equation, the regular long-wave equation, and the KdV-Burgers equation are solved by a geometric lattice factorization algorithm based on isolator theory and conformal computing systems. From the exact solutions of the rod equation, the periodic wave becomes an anticlockwise isolated wave when $c < 0$, $g_1(c) < g < g_2(c)$, φ_0 satisfies $\varphi_-^* < \varphi_0 < \varphi_+^*$ and φ_0 converges to φ_-^* . From the exact solutions of the regular long-wave equation and the KdV-Burgers equation, different parameter forms can be obtained by the factorization algorithm. It is thus shown that the use of geometric grid factorization algorithms enables the exact solution of non-linear mathematical physics equations and provides a new research direction for the development of the engineering field.

1. Introduction

The solution of nonlinear mathematical physics equations is a topical problem in nonlinear science, and both exact and approximate solutions of nonlinear mathematical physics equations are of great importance [1–2]. Their importance for practical problems is reflected in the assessment of the reliability of numerical simulations of real models and the interpretation of actual physical phenomena, and thus have been widely studied, and the development of soliton theory illustrates the importance of constructing exact solutions to nonlinear mathematical physics equations [3–5].

As the exact solution is a very important part of the theory of isolines, it has been studied by many scientists, including mathematicians, physicists, and experts in engineering [6–7]. After years of continuous efforts, many experts and scholars in this field at home and abroad have proposed a series of methods for constructing exact solutions. Such as the elliptic function expansion method, the Hirota bilinear method, the Backlund transformation method, the chi-squared method, the trial function method, the formal variable

Corresponding author: Xiaohan Yuan, mail address: xhyuan020801@163.com

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Corresponding author.

separation method, the mixed exponential method, the geometric method and various algebraic expansion methods [8–10]. The emergence of various solution methods has not only led to the solution of equations that were previously difficult to solve but also to the discovery of new solutions of physical importance, giving rise to an endless stream of momentum [11–12].

Different methods can be used to achieve solutions to mathematical physics equations. In the literature [13], the characteristics of the Hirota linearity are used to analyze the arguments for set block solutions and set block type solutions of mathematical physics equations, and 3D maps are drawn through explicit solutions of mathematical physics equations, which in turn reflect the dynamics of Hirota applied to the solution of mathematical physics equations. The specific properties of Klein solutions of quadratic norm-invariant nonlinear equations are investigated in the literature [14], where solutions of the two-dimensional Gordon equation are solved by asymptotic profiles, and suitable approximate solutions of this equation are given based on the nonlinear Fourier series. The literature [15] analyses the wave solutions of the Peyrard-Bishop model DNA flow extraction closed form based on the (G'/G) expansion method, by which some graphs of this model can be effectively obtained as a means of analyzing the dynamic behavior of the waves.

In addition, the literature [16] analyses the time-dependent damping of the Eulerian system of equations by varying the time-asymptotic damping so that the properties of the Eulerian system adhere to the time-asymptotic formation of the diffusion equation and shows that the time weighting of its global solution depends on the parameter settings. The literature [17] presents a new derivation for the solution of the Gibbs-Duhem equation for partial differential G and partial differential ξ through mathematics and thermodynamics, allowing it to show different solutions through different perspectives. The literature [18] analyzed the optical soliton and single-row wave solutions of the Triki-Biswas equation, and the soliton and single-row wave solutions of this equation can be effectively achieved with polynomial complete discriminant algorithms, enabling its research direction to be extended.

In order to investigate the symbolic computation of exact solutions to non-linear mathematical, physical equations, this paper first addresses the theory of isolines and symbolic computational systems, including specific forms of common isoline waves and specific applications of differential equations. Then the geometric grid factorization algorithm is illustrated, with the factorization preprocessing of triangular grid matrices and square grid matrices. Finally, exact solutions are solved for several classes of non-linear mathematical and physical equations, including periodic exact traveling wave solutions of the rod equation and exact solutions of the regular long-wave equation and the KdV-Burgers equation. From the exact solutions, it can be seen that the use of the geometric grid factorization algorithm can be effective in solving the exact solutions of the non-linear mathematical physics equations, providing a new research direction for solving problems in the engineering field.

2. Isolaton Theory and Symbolic Computing Systems

As scientific research continues to advance, many natural science and engineering technology problems eventually boil down to non-linear partial differential equations, making it particularly important to solve exact solutions of non-linear partial differential equations. In general, the study of the correlation properties of isolines usually translates into the study of the solutions of their corresponding nonlinear mathematical physics equations. In recent years, the study of exact traveling wave solutions of nonlinear mathematical physics equations, in particular soliton equations, has attracted considerable scholarly attention. This chapter focuses on the role of soliton theory and symbolic computing systems for solving exact solutions of non-linear mathematical physics equations as a means of providing a theoretical basis for the exact traveling wave solutions to be carried out later.

2.1. Isolator Theory

2.1.1. Isolated Waves

The term isolated wave was initially coined by the British scholar Russell from a strange phenomenon he noticed in 1834, where a fast-moving boat was on a small river, and when the boat suddenly stopped,

the water waves driven by the boat formed a smooth cluster of water in the direction the boat was moving and departed rapidly forward. As it moved forward, the shape and speed of the water waves remained essentially as they had been until perhaps one to two kilometers away when they gradually disappeared, after which he named the peculiar water waves as isolated waves.

Russell later explored this magical mass of water extensively, using the flume as a river to recreate the situation, and was lucky enough to discover isolated waves again, whereupon Russell ventured to speculate that this particular water wave was a special solution to a class of physical equations.

Later on, another researcher proposed the KdV equation based on the study of isolated waves and found that when two isolated waves collide, they do not affect each other but keep their original shapes and continue to move forward, and can maintain their original velocities. In other words, the nature of the two isolated waves does not change before and after the collision. This study also shows that isolated waves have a certain degree of stability, hence the term isolator. Isolines have attracted a great deal of interest from scientists because of their special properties.

2.1.2. Common Single Isolated Wave Solutions

The theory of solitons plays an important role not only in physics but also in mathematics, biology, and chemistry. From the analysis of mathematical disciplines, an isolator refers to a special kind of solution of a non-linear evolution equation. Since the discovery of isolators, many isolated wave solutions of nonlinear evolution equations have been found, including the most common KdV equation, KP equation, Burgers equation, Boussinesq equation, etc. For the shape of the isolated waves, the common single isolated wave solutions are shown in Figure 1.

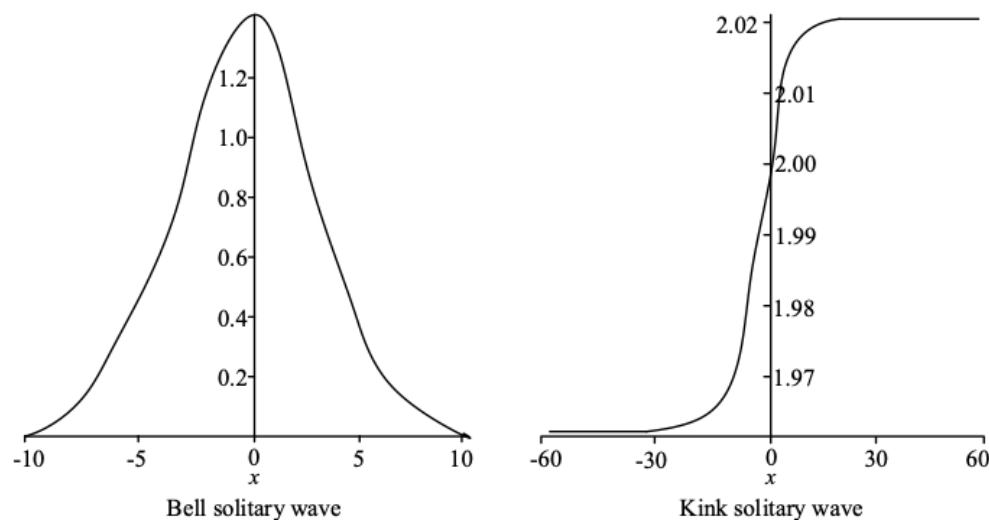


Figure 1: Common single isolated wave solutions

In addition to these two types, there are also anti-bell isolated waves, which are the opposite of bell isolated waves, and anti-kink isolated waves, which are the opposite of kink isolated waves. The different waveforms represent different types of isolated wave solutions to the equations of nonlinear mathematical physics.

2.2. Symbolic Computing Systems

Mathematical computing includes not only numerical computing but also symbolic computing, called computer algebra. Literally and simply understood, symbolic computation is the performance of operations on certain symbols representing mathematical objects. These symbols include numbers, polynomials, rational functions, matrices, etc. [19].

For simple calculations, we can use paper and pencil for manual calculations, but for complex calculations, we have to rely on computers. In recent years, the field of computing has risen to new heights, and symbolic computing systems have emerged, which in many cases, can replace bulky human labor. This section focuses on some basic commands for solving non-linear mathematical physics equations with the Maple symbolic computing system.

2.2.1. Differential and Differential Equations

(1) Differentiation

In the process of solving a non-linear model, the differentiation command is often used. In Maple, the `diff`, `implicitdiff`, and `D` commands can be used to calculate differentiation, and this section focuses on the `diff` and `D` commands.

The `diff` command acts on expressions and can be used to differentiate a single variable and multiple variables separately, for example, as follows:

$$> \text{diff}(f, [x1\$n]); \quad \frac{d^n}{dx_1^n} f \quad (1)$$

$$> \text{diff}(f, x1, \dots, xj); \quad \frac{d^j}{dx_j \cdots dx_1} f \quad (2)$$

$$> \text{diff}(f, x1\$n, [x2\$n, x3], \dots, xi, [xk\$m]); \quad \frac{d^r}{dx_k^m \cdots dx_j \cdots dx_3 dx_2^n dx_1^n} f \quad (3)$$

The `implicitdiff` command is mainly used to differentiate implicit functions, which are generally determined by equation $f(x, y) = 0$. The usage is `implicitdiff(f, y, x)`, which evaluates to $\frac{dy}{dx}$, for example

$$> \text{implicitdiff}(x^2 + y^2 = 1); \quad -\frac{x}{y} \quad (4)$$

The `D` command usually acts on functions rather than expressions, as an example:

$$> f := x^2 + 3*x; \quad f := x \rightarrow x^2 + 3x \quad (5)$$

$$> D(f); \quad x \rightarrow 2x + 3 \quad (6)$$

(2) Differential equations

When solving partial differential equations, we generally convert them to ordinary differential equations using the traveling wave transform. In Maple, the `dsolve` command can be used to calculate the ordinary differential model, commonly used as `dsolve(ODE, y(x))`, where `ODE` denotes an ordinary differential equation, for example, considering the following differential equation:

$$\frac{dy}{dx} = x + y \quad (7)$$

$$> \text{dsolve}(\text{diff}(y(x), x) = x + y(x), y(x)); \quad y(x) = -x - 1 + e^x _C1 \quad (8)$$

The solution obtained here is the general solution of the equation, where `_C1` is a constant term, and given a special value of `_C1`, the special solution of the equation is obtained.

2.2.2. 2D and 3D Graphics

Maple's graphing functions allow you to plot two and three-dimensional graphs of exact solutions to equations. The command to plot two-dimensional graphs in Maple is `plot`, which takes two parameters, the function and the area to be plotted, and is used in `plot(f(x), x = a, b)`, where x is the independent variable, $f(x)$ is a function of x and a, b is a range of values for x .

The command for drawing 3D shapes in Maple is `plot3d`, which is used in a similar way to `plot`, e.g. `plot3d(f(x, y), x = a, b, y = c, d)`, where $f(x, y)$ is a function on x and y , a, b, c, d are constants, a, b is a range of values for x and c, d is a range of values for y .

3. Geometric Grid Factorisation Algorithm

For the conformal calculation of exact solutions of non-linear mathematical physics equations, in addition to understanding isolated sub-theory and the symbolic computational system in which the analysis is carried out, it is necessary to make the solutions of non-linear mathematical physics equations decoupled to the relevant sub-blocks by means of a latticework scale, thus enabling the solution of exact solutions of non-linear mathematical physics equations.

3.1. Factorisation Pre-processing of Triangular Lattice Matrices

Given a k -layer triangular discrete structured mesh centered at the origin, the total number of nodal degrees of freedom is easily known to be

$$\dim(T_{Gridk}) = \frac{1}{2}(3k+1)(3k+2) \quad (9)$$

3.1.1. Grid Factorisation Pre-processing

Theorem 1

$$P_{GT1} \cdot GT1 = \text{Diag}(d_{GT1}) \cdot P_{GT1} \quad (10)$$

Where the eigenmatrix P_{GT1} has exactly three non-zero elements in each row except the first, all originating from the third root of 1. Then

$$P_{GT1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & -\omega & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & -\omega & 0 & 1 & 0 \\ 0 & -\omega^{-1} & -\omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega & 0 & -\omega^{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega & 0 & -\omega^{-1} & 0 & 1 & 0 \\ 0 & -\omega & -\omega^{-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (11)$$

$$d_{GT1} = \{1, 1, 1, 1, -\omega^{-1}, -\omega^{-1}, -\omega^{-1}, -\omega, -\omega, -\omega\} \quad (12)$$

Further, the eigenmatrix P_{GT1} can be divided into three orthogonal submatrices (rows 1–4, rows 5–7, rows 8–10), then

$$P_{JG10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad (13)$$

$$P_{GI10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & -\omega & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & -\omega & 0 & 1 & 0 \\ 0 & -\omega^{-1} & -\omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (14)$$

$$P_{GII10} = P_{GI10}^* \quad (15)$$

Let A' denote the transpose of A , A^* the complex coprime of A and A^H the complex coprime transpose of A .

Lemma 1 The following mutually orthogonal equations hold in the ordinary sense

$$P_{GI10} \cdot P'_{JG10} = P_{GII10} \cdot P'_{JG10} = 0_{3 \times 4}, \quad P_{GI10} \cdot P_{GII10}^H = 0_{3 \times 3} \quad (16)$$

At this point, the triangular discrete structured mesh preprocessing matrix framework has been established and can be applied to discrete mathematical physics equation solvers, such as the generalized eigenvalue problem for the zero-sided value condition of the Laplace equation for positive triangles using linear elements, viz:

$$A_{T1}u = \lambda B_{T1}u \quad (17)$$

Of which

$$A_{T1} = \begin{pmatrix} 6 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 6 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 6 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 6 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 6 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 6 & -1 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 6 \end{pmatrix} \quad (18)$$

$$B_{T1} = 12I_{10} - A_{T1} \quad (19)$$

I_n denotes a unit matrix of order n .

3.1.2. Scalable Algorithms for Triangular Discrete Structured Networks

Theorem 2

$$P_{GT2} \cdot GT2 = \text{Diag}(d_{GT2}) \cdot P_{GT2} \quad (20)$$

where the eigenmatrix P_{GT2} has exactly three non-zero elements in each row except the first, all originating from the third root of 1.

Taking a 25-dimensional eigenpolynomial factorization as an example, we need only verify that

$$\det(A_{PJ25} - \lambda B_{PJ25}) = C_P(\lambda + 1)(15\lambda^3 + 40\lambda^2 + 17\lambda + 1)(35\lambda^3 + 179\lambda^2 + 255\lambda + 113)^2 \quad (21)$$

Is a factor of the overall characteristic polynomial $\det(A_{T2} - \lambda B_{T2})$, where C_P is a constant and (A_{T2}, B_{T2}) is the stiffness matrix and mass matrix respectively. Then the complex part of $\det(A_{T2} - \lambda B_{T2})$ is:

$$\frac{\det(A_{T2} - \lambda B_{T2})}{\det(A_{PJ25} - \lambda B_{PJ25})} = \frac{1}{2^{45}} p_7(\lambda)^2 \quad (22)$$

$$p_7(\lambda) = 703459\lambda^7 + 7647417\lambda^6 + 35633772\lambda^5 + 92948820\lambda^4 + 148664826\lambda^3 + 150035742\lambda^2 + 94597500\lambda + 35467812 \quad (23)$$

And $p_7(\lambda)$ is exactly equal to the other two factors of the factorization, then:

$$A_{GI25} = P_{GI25} \cdot A_{T2} \cdot P_{GI25}^H \quad (24)$$

$$B_{GI25} = 36I_7 - A_{GI25} \quad (25)$$

$$\det(A_{T2} - \lambda B_{T2}) = C_R \det(A_{PJ25} - \lambda B_{PJ25}) (\det(A_{GI25} - \lambda B_{GI25}))^2 \quad (26)$$

Where the constant C_R is a constant and the notation A^H denotes the complex common transpose of A .

The above algorithm gives a factorisation algorithm for a three-block characteristic polynomial, if restricted to factorisation in the real range, when a matrix of order 25 intelligently decomposes two real factors with uneven parallel loadings.

3.2. Square Discrete Structured Grid Factorisation Pre-processing

Consider the generalized eigenvalue problem of solving the square Laplace equation with zero-sided value conditions using linear elements $Au = \lambda Bu$, and note that the degrees of freedom of a discrete structured grid with symmetric centroid quadrilateral S_4^k are

$$\dim(S_4^k) = (2k + 1)^2 \quad (27)$$

When $k = 2$, $\dim(S_4^2) = 11$, note P_3, J_3, GI_3 is the mesh node transformation matrix at this point, and AR_1 is the stiffness matrix formed by the Laplace linear element with zero boundary conditions on that uniform mesh node.

The corresponding matrix diagram is given by

$$P_3 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}, J_3 = \begin{bmatrix} 2 & 2 & -1 \\ -2 & 0 & -2 \\ -1 & 2 & 1 \end{bmatrix}, GI_3 = \begin{bmatrix} 1 & 2 & -i1 \\ i2 & 0 & -i2 \\ i1 & -2 & -1 \end{bmatrix} \quad (28)$$

The four-cycle condition satisfied $R_{11}^4 = I_{11}$ is equivalent in the domain of complex numbers to:

$$(I_{11} - R_{11}) \cdot (I_{11} + R_{11}) \cdot (iI_{11} - R_{11}) \cdot (iI_{11} + R_{11}) = 0 \quad (29)$$

It is not difficult to verify that the corresponding four subspace bases of Matrix Diagram P_4, J_4, GI_4 can be:

$$PR_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (30)$$

$$JR_2 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \quad (31)$$

$$GI_2 = \begin{pmatrix} -1 & 0 & i & 0 & 0 & 0 & -i & 0 & 1 \\ 0 & -1 & 0 & -i & 0 & i & 0 & 1 & 0 \end{pmatrix} \quad (32)$$

$$GII_2 = GI_2^* \quad (33)$$

The eigenvalue algorithm for the calculation of the corresponding four blocks of eigenpolynomials is a set of decomposition algorithms equivalent to the original 11th order edge calculation. This gives us a set of four block decomposition algorithms for solving generalized eigenvalues $AR_1u = \lambda BR_1u$, $BR_1 = 8I_{10} - AR_1$.

This decomposition algorithm can be applied directly to the stiffness matrix formed by the same degrees of freedom at the boundary of a Laplace linear element cycle, i.e:

$$AR_3 = \begin{pmatrix} 4 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 4 \end{pmatrix} \quad (34)$$

The reason is that, like AR_1 , AR_3 is a pairwise easy matrix of R_{11} , i.e:

$$AR_3 \cdot R_{11} = R_{11} \cdot AR_3 \quad (35)$$

3.2.1. Scalability of Square Discrete Structured Mesh Substrates

When $k = 3$, the square discrete structured lattice degrees of freedom can be expressed as $\dim(S_4^3) = 36$, which corresponds to the matrix plotted as

$$PG_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 5 & 6 & 7 & 7 & 6 & 2 \\ 4 & 5 & 6 & 6 & 5 & 3 \\ 3 & 5 & 6 & 6 & 5 & 4 \\ 2 & 6 & 7 & 7 & 6 & 5 \\ 1 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \quad (36)$$

The stiffness matrix for PG_6 is shown above, and the matrix for $JG_6, GIG_6, GIIG_6$ is omitted here.

From this, we obtain the following algorithm for solving a set of four block decompositions of the generalized characteristic:

$$AR_3u = \lambda BR_3u, \quad BR_3 = 8I_{36} - AR_3 \quad (37)$$

$$PAR_7 = PR_5 \cdot AR_3 \cdot PR'_5 = \begin{pmatrix} 16 & -4 & 0 & -4 & 0 & 0 & 0 \\ -4 & 16 & -4 & 0 & -4 & 0 & 0 \\ 0 & -4 & 16 & -4 & 0 & -4 & 0 \\ -4 & 0 & -4 & 16 & -4 & 0 & 0 \\ 0 & -4 & 0 & -4 & 16 & -8 & 0 \\ 0 & 0 & -4 & 0 & -8 & 16 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 & 4 \end{pmatrix} \quad (38)$$

$$PBR_7 = PR_5 \cdot BR_3 \cdot PR'_5 = \begin{pmatrix} 16 & 4 & 0 & 4 & 0 & 0 & 0 \\ 4 & 16 & 4 & 0 & 4 & 0 & 0 \\ 0 & 4 & 16 & 4 & 0 & 4 & 0 \\ 4 & 0 & 4 & 16 & 4 & 0 & 0 \\ 0 & 4 & 0 & 4 & 16 & 8 & 0 \\ 0 & 0 & 4 & 0 & 8 & 16 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 \end{pmatrix} \quad (39)$$

$$JAR_6 = JR_5 \cdot AR \cdot JR'_5 = \begin{pmatrix} 16 & -4 & 0 & 4 & 0 & 0 \\ -4 & 16 & -4 & 0 & -4 & 0 \\ 0 & -4 & 16 & -4 & 0 & -4 \\ 4 & 0 & -4 & 16 & 4 & 0 \\ 0 & -4 & 0 & 4 & 16 & 0 \\ 0 & 0 & -4 & 0 & 0 & 16 \end{pmatrix} \quad (40)$$

$$GAR_6 = GI_2 \cdot AR_5 \cdot GI_2^H = \begin{pmatrix} 16 & -4 & 0 & 4i & 0 & 0 \\ -4 & 16 & -4 & 0 & -4 & 0 \\ 0 & -4 & 16 & -4 & 0 & -4 \\ -4i & 0 & -4 & 16 & -4i & 0 \\ 0 & -4 & 0 & 4i & 16 & -4 + 4i \\ 0 & 0 & -4 & 0 & -4 - 4i & 16 \end{pmatrix} \quad (41)$$

$$JBR_6 = JR_5 \cdot BR_5 \cdot JR'_5 = \begin{pmatrix} 16 & 4 & 0 & -4 & 0 & 0 \\ 4 & 16 & 4 & 0 & 4 & 0 \\ 0 & 4 & 16 & 4 & 0 & 4 \\ -4 & 0 & 4 & 16 & -4 & 0 \\ 0 & 4 & 0 & -4 & 16 & 0 \\ 0 & 0 & 4 & 0 & 0 & 16 \end{pmatrix} \quad (42)$$

$$GBR_6 = GI_2 \cdot BR_5 \cdot GI_2^H = \begin{pmatrix} 16 & 4 & 0 & -4i & 0 & 0 \\ 4 & 16 & 4 & 0 & 4 & 0 \\ 0 & 4 & 16 & 4 & 0 & 4 \\ 4i & 0 & 4 & 16 & 4i & 0 \\ 0 & 4 & 0 & -4i & 16 & 4 - 4i \\ 0 & 0 & 4 & 0 & 4 + 4i & 16 \end{pmatrix} \quad (43)$$

Theorem 1 The 36th order characteristic polynomial $\det(A_{T36} - \lambda B_{T36})$ is now decomposed into a parallel-loaded uniform four-factor product as

$$\det(AR_3 - \lambda BR_3) = C \det(PAR_6 - \lambda PBR_6) \cdot \det(JAR_6 - \lambda JBR_6) \cdot (\det(GAR_6 - \lambda GBR_6))^2 \quad (44)$$

where C is a constant.

3.2.2. Graphical Representation and Scalability of the Algorithm

When $k = 3$, the square discrete structured lattice degrees of freedom can be expressed as $\dim(S_4^3) = 36$, and the positions of exactly four non-zero elements in each row of the four sets of bases can be seen in the 6×6 matrix schematic above.

It is easy to see that this contact diagram, which reflects the symmetry of the different bases, is scalable, i.e. if when $k = 4$, then $n = 49$, and PG_7 is expanded by adding a layer of edges to the matrix as

$$PG_7 = \begin{pmatrix} 12 & 11 & 8 & 7 & 9 & 10 & 12 \\ 10 & 6 & 5 & 4 & 3 & 6 & 11 \\ 9 & 3 & 2 & 1 & 2 & 5 & 8 \\ 7 & 4 & 1 & 0 & 1 & 4 & 7 \\ 8 & 5 & 2 & 1 & 2 & 3 & 9 \\ 11 & 6 & 3 & 4 & 5 & 6 & 10 \\ 12 & 10 & 9 & 7 & 8 & 11 & 12 \end{pmatrix} \quad (45)$$

In turn, a new round of basis functions can be formed by extending the contact matrix diagram, e.g. a 7×7 matrix directly generates a 21×49 basis matrix PR_7 with a transposed JR_7 20×49 basis matrix. Similarly, GI_6 is extended by adding a layer of edges to GI_7 . It is easy to verify that these three sets of matrices satisfy the orthogonal basis conditions required for the problem as

$$PR_7 \cdot (I_{49} - R_{49}) = 0, \quad JR_7 \cdot (I_{49} + R_{49}) = 0 \quad (46)$$

$$GI_7 \cdot (iI_{49} - R_{49}) = 0, \quad GII_7 \cdot (iI_{49} + R_{49}) = 0 \quad (47)$$

The scalable algorithm for the liaison diagram is actually determined by a kind of multicolour ordering of the structured grid, an algorithm that is easy to implement through programmatic optimization.

For analytical and programming convenience, the non-zero elements can also be written as follows:

$$PM(m, k, j) := \begin{pmatrix} \{k, j\} & \{j, -k + m + 1\} \\ \{m + 1 - j, k\} & \{m + 1 - k, m + 1 - j\} \end{pmatrix} \quad (48)$$

$$JM(m, k, j) := \begin{pmatrix} \{k, j\} & -\{j, -k + m + 1\} \\ -\{m + 1 - j, k\} & \{m + 1 - k, m + 1 - j\} \end{pmatrix} \quad (49)$$

$$GI(m, k, j) := \begin{pmatrix} -\{k, j\} & i\{j, m + 1 - k\} \\ -i\{m + 1 - j, k\} & \{m + 1 - k, m + 1 - j\} \end{pmatrix} \quad (50)$$

In turn, we find the following three sparse matrices for the given parameter (n, j, k) of the square grid: $PM_m(n, j, k)$, $JM_m(n, j, k)$, $GM_m(n, j, k)$.

The column positions of the four non-zero elements in each row are the same, so the distribution of the four non-zero elements in the columns of the sparse matrices by row can be found.

Given the square grid parameter (n, j, k) , the $PM_m(n, j, k)$, $JM_m(n, j, k)$, $GM_m(n, j, k)$ column distribution of the three sparse matrices by the four non-zero elements of the rows is given by

$$PM_m(n, k, j) = \{n(k-1) + j, n(j-1) + n + 1 - k, n(n-j) + k, n(n-k) + n + 1 - j\} \quad (51)$$

$$\begin{aligned} JM_m(n, k, j) &= \{n(k-1) + j, -(n(j-1) + n + 1 - k), -n(n-j) + k, n(n-k) + n + 1 - j\} \\ GM_I(n, k, j) &= \{-(n(k-1) + j), i(n(j-1) + n + 1 - k), -i(n(n-j) + k), n(n-k) + n + 1 - j\} \\ GM_{II}(n, k, j) &= GM_I(n, k, j)^H \end{aligned} \quad (52)$$

The notation i used in this algorithm denotes imaginary units, and the notation H denotes complex conjugates.

4. Exact Solutions of Several Classes of Non-linear Mathematical Physics Equations

Based on the previous analysis of methods for solving exact solutions of mathematical and physical equations, this chapter focuses on periodic traveling wave solutions and exact solutions of the rod equation and the KdV-Burgers equation.

4.1. Periodically Exact Traveling Wave Solutions of the Rod Equation

The specific expressions for the rod equation are as follows:

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}) \quad (53)$$

This equation can be analyzed by geometric lattice factorization for traveling waves, which in turn leads to traveling wave solutions for this family of rod equations, including isolated wave solutions, periodic excitation solutions, etc.

4.1.1. Theorems and Corollaries for the Rod Equation

Theorem 1 For a given parameter $\gamma < 0$, let c, g, φ_0 be a constant and $\xi = x - ct$, $\sin^{-1}(\cdot)$ be the inverse of the trigonometric function $\sin(\cdot)$, $sn^{-1}(\cdot, \cdot)$ be the inverse of the elliptic function $sn(\cdot, \cdot)$, $\Pi(\cdot, \cdot, \cdot)$ be a type III elliptic integral and $K(\cdot)$ be a type I elliptic integral.

Corollary 1 If parameter $\gamma < 0$, then the bar equation has two periodic wave solutions as follows:

$$u_1(\xi) = \frac{(\gamma - 1)c}{2\gamma} + A \cos \frac{\xi}{\sqrt{-\gamma}} \quad (54)$$

$$u_2(\xi) = \frac{(\gamma - 1)c}{2\gamma} + A \sin \frac{\xi}{\sqrt{-\gamma}} \quad (55)$$

Here $A \neq 0, c \neq 0$ is an arbitrary constant and $\xi = x - ct$.

Corollary 2 If parameter $\gamma < 0$, then the rod equation has the following complex display solution:

$$u(\xi) = \frac{(\gamma - 1)c}{2} + Ae^{i\xi/\sqrt{-\gamma}} \quad (56)$$

Here A, c is an arbitrary constant and $\xi = x - ct$.

Next, we will prove the above theorem and corollary. Before proving again, we plot the planes of the implicit periodic wave solution and the explicit periodic wave solution so that we can compare them later.

Example 1 Let $\gamma = -3, c = -6$, then $q = 3, g_1(c) = -15, g_2(c) = -2$. Let $g = -3$, we get $\varphi_-^* = -5.23141, \bar{\varphi}_+ = -4.76854$. Here let φ_0 be $-4.82, -5.31, -5.23142$ respectively, thus also showing that $c < 0, g_1(c) < g < g_2(c)$

and $\varphi_-^* < \varphi_0 < \bar{\varphi}_+$, so for the corresponding initial values φ_0 , the rod equation has periodic wave solutions $u = \varphi(\xi)$. One of these is the implicit function expression, and using Maple software, the image of the implicit function $u = \varphi(\xi)$ can be obtained, as shown in Figure 2.

From the image of the implicit function $u = \varphi(\xi)$, it can be seen that when $c < 0, g_1(c) < g < g_2(c)$, φ_0 satisfies $\varphi_-^* < \varphi_0 < \bar{\varphi}_+$ and φ_0 converges to φ_-^* , the periodic wave becomes an anticlockwise isolated wave.

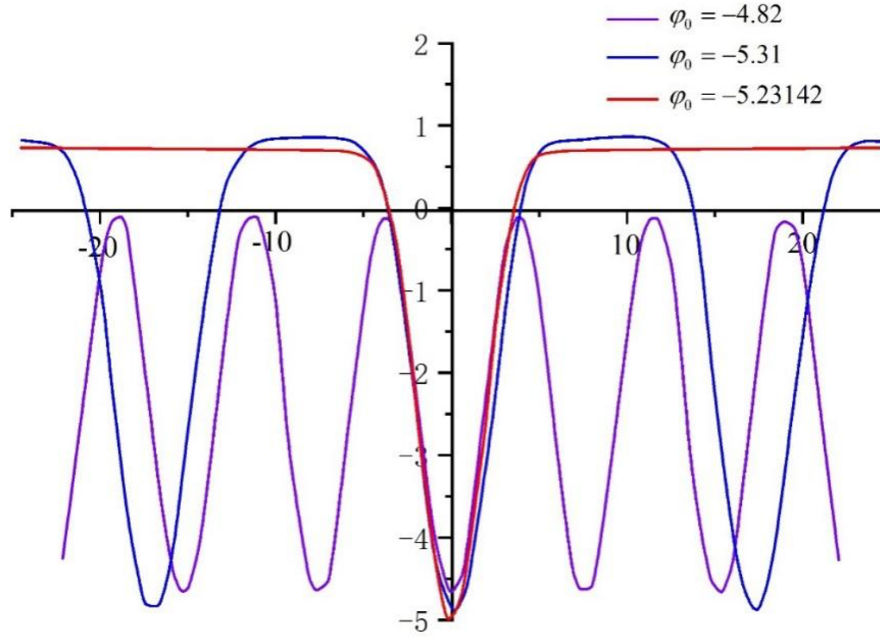


Figure 2: Image of the implicit function

4.1.2. Proof of Theorems and Corollaries for the Rod Equation

The orbit through $(\varphi_0, 0)$ is a closed orbit when φ_0 satisfies one of the conditions in Theorem 1. Assume that its expressions in the parametric form are $u = \varphi(\xi)$ and $y = y(\xi)$. From the above derivation, we know that $u = \varphi(\xi)$ is a periodic wave solution of the rod equation. In the following, we will prove the relevant conclusions of Theorem 1 and the corollaries respectively.

Conclusion 1 Let $\Gamma_{\varphi_0}^{**}$ be the closed orbit through $(\varphi_0, 0)$ and obtain its distribution as shown in Figure 3. Then in the phase plane $\varphi - y$, it has the following expression:

$$y = \pm \frac{1}{\sqrt{-\gamma}} \sqrt{\frac{(m - \varphi)(n - \varphi)(\varphi - \varphi_0)}{q - \varphi}}, \quad \varphi_0 \leq \varphi \leq n < m < q \quad (57)$$

Letting his period be $2T_1$ and $\varphi(0) = \varphi_0$, we obtain $\varphi(T_1) = \varphi(-T_1) = n$.

Taking equation (57) into $\frac{d\varphi}{d\xi} = y$ and integrating along $\Gamma_{\varphi_0}^{**}$, we have

$$\int_{\varphi_0}^n \sqrt{\frac{q - s}{(m - s)(n - s)(s - \varphi_0)}} ds = \frac{1}{\sqrt{-\gamma}} (T_1 - |\xi|) \quad (58)$$

Here $\varphi_0 \leq \varphi < n < m < q$.

Noting $\varphi(0) = \varphi_0$, completing the integral above leads to conclusion 1.

Conclusion 2 Let $\Gamma_{\varphi_0}^{00}$ be a closed rail through $(\varphi_0, 0)$, the distribution of which is shown in Figure 4. Thus, in the phase plane $\varphi - y$, it is expressed as:

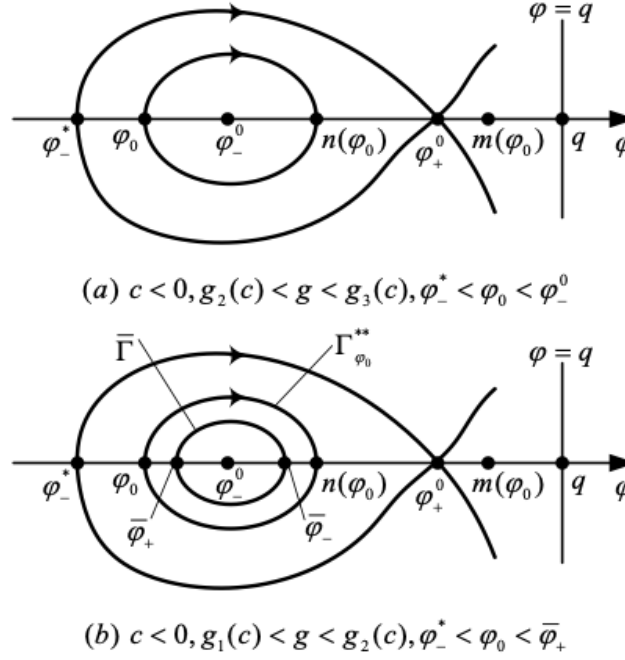


Figure 3: Diagrammatic proof of conclusion 1

$$y = \pm \frac{1}{\sqrt{-\gamma}} \sqrt{\frac{(\varphi_0 - \varphi)(\varphi - m)(\varphi - n)}{\varphi - q}}, \quad q < n < m \leq \varphi \leq \varphi_0 \quad (59)$$

Letting his period be $2T_1$ and $\varphi(0) = \varphi_0$, we obtain $\varphi(T_2) = \varphi(-T_2) = m$.

Taking equation (59) into $\frac{d\varphi}{d\xi} = y$ and integrating along $\Gamma_{\varphi_0}^{00}$, we have:

$$\int_m^\varphi \sqrt{\frac{s - q}{(\varphi_0 - s)(s - m)(s - n)}} ds = \frac{1}{\sqrt{-\gamma}} (T_2 - |\xi|) \quad (60)$$

Here $q < n < m < \varphi \leq \varphi_0$.

Noting $\varphi(0) = \varphi_0$, completing the integral above leads to conclusion 2.

4.2. Exact Solutions of the Regular Long-wave Equation

The regular long-wave equation, originally proposed as a modified model of the KdV equation, occupies an important place in the study of isolator theory and has attracted much attention as it can be used to describe many physical phenomena, such as wave motion, ion motion, and non-linear dispersive waves.

$$u_t + u_x + \lambda uu_x - \mu u_{xxt} = 0 \quad (61)$$

Performing the traveling wave transformation $u = u(\xi)$, $\xi = kx + \omega t$ on equation (61) yields the following ordinary differential equation with only a single variable ξ after the transformation:

$$\omega u' + ku' + \lambda ku u' - \mu k^2 \omega u'' = 0 \quad (62)$$

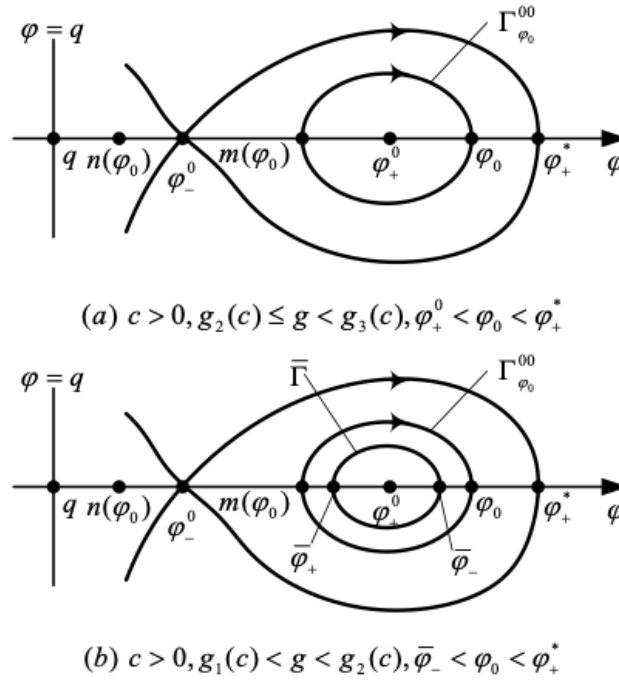


Figure 4: Diagrammatic proof of conclusion 2

The above equation can be further simplified by integration by setting the integration constant to zero, which gives:

$$\omega u + ku + \frac{1}{2}\lambda ku^2 - \mu k^2 \omega u' = 0 \quad (63)$$

Let the above equation have a solution of the form $u(\xi) = \sum_{i=0}^p a_i e^{i\xi} / \sum_{j=0}^q b_j e^{j\xi}$. By balancing the non-linear term and the highest order derivative term in equation (63) we can obtain $p = q$ in the solution, here let $p = q = 1$, then the solution of equation (63) can be expressed as

$$u(\xi) = \frac{a_0 + a_1 e^\xi}{b_0 + b_1 e^\xi} \quad (64)$$

Bringing equation (64) into equation (63), combining all the same power terms of e^ξ and setting the coefficients of $e^{i\xi}$ to zero gives the following system of algebraic equations, i.e:

$$\begin{aligned} e^{0\xi} : 2\omega a_0 b_0^2 + 2ka_0 b_0^2 + \lambda ka_0^2 b_0 &= 0 \\ e^{1\xi} : 4\omega a_0 b_0 b_1 + 2\omega a_1 b_0^2 - 2\mu k^2 \omega a_1 b_0^2 + 2\lambda ka_0 a_1 b_0 + 4ka_0 b_0 b_1 \\ &\quad + 2\mu k^2 \omega b_1 a_0 b_0 + 2ka_1 b_0^2 + \lambda ka_0^2 b_1 = 0 \\ e^{2\xi} : 2\omega a_0 b_1^2 + 2\lambda ka_0 a_1 b_1 + 4\omega a_1 b_0 b_1 + 4ka_1 b_0 b_1 + 2\mu k^2 \omega a_1 b_0 b_1 \\ &\quad + \lambda ka_1^2 b_0 + 2ka_0 b_1^2 - 2\mu k^2 \omega b_1^2 a_0 = 0 \\ e^{3\xi} : 2\omega a_1 b_1^2 + 2ka_1 b_1^2 + \lambda ka_1^2 b_1 &= 0 \end{aligned} \quad (65)$$

Solving equation (65) with the help of Maple gives:

$$\omega = -\frac{1}{2} \frac{k(2b_1 + \lambda a_1)}{b_1}, \quad a_0 = \frac{a_1 b_0}{b_1} \quad (66)$$

Substituting the result of Eq. (66) into Eq. (64), combined with $u = u(\xi)$, $\xi = kx + \omega t$, yields the exact solution of the RLW equation as:

$$u(x, t) = \frac{\frac{a_1 b_0}{b_1} + a_1 e^{kx - \frac{1}{2} \frac{k(2b_1 + \lambda a_1)t}{b_1}}}{b_0 + b_1 e^{kx - \frac{1}{2} \frac{k(2b_1 + \lambda a_1)t}{b_1}}} \quad (67)$$

For Eq. (67), if the constant in Eq. is taken to be $a_1 = 12, b_0 = 1, b_1 = 20, k = 0.5, \lambda = 0.5$, where x, t takes on a range of values of $x \in [-10, \dots, 10], t \in [2, \dots, 5]$, the three-dimensional diagram of its exact solution is shown in Fig. 5.

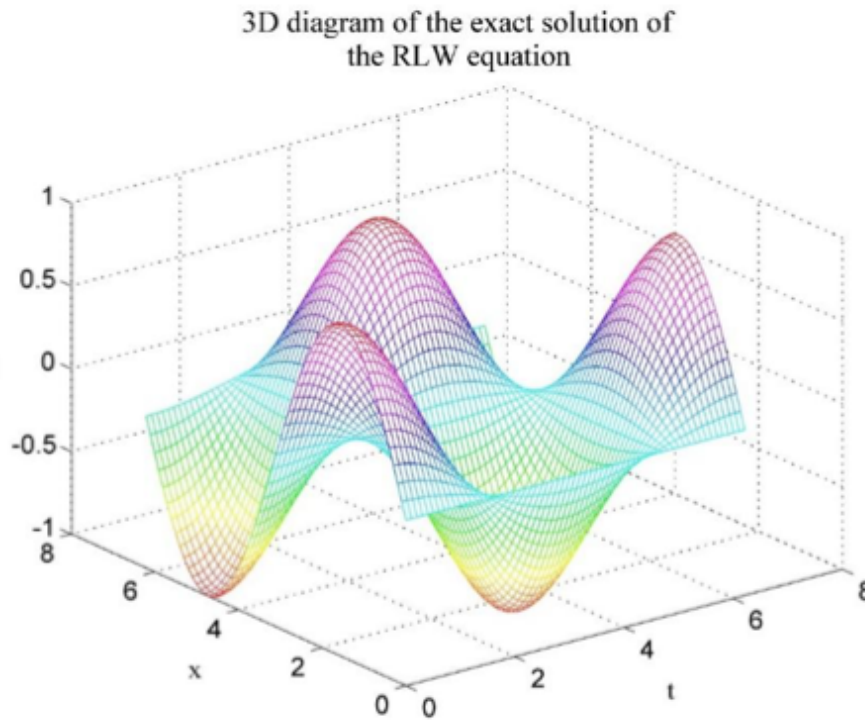


Figure 5: 3D diagram of the exact solution of the RLW equation

4.3. Exact Solutions of the KdV-Burgers Equation

The KdV-Burgers equation describes the propagation of finite amplitude waves in a dispersive medium and degenerates to the KdV equation when relaxation effects are not taken into account and to the Burgers equation when dispersion effects are not taken into account. The KdV-Burgers equation is also an important equation in physics and fluid mechanics that describes the flow of bubble-containing liquids in elastic tubes. The KdV-Burgers equation is also an important equation in physics and hydrodynamics, describing the flow of bubble-laden liquids, liquid flow in elastic tubes, and non-linear waves in plasma. In this paper, the KdV-Burgers equation is solved based on a geometric grid factorisation algorithm, and its exact solution is obtained.

The following KdV-Burgers equation is considered, i.e:

$$u_t + uu_x - \alpha u_{xx} + \beta u_{xxx} = 0 \quad (68)$$

Where α, β is the dissipation and dispersion coefficients respectively.

By introducing the traveling wave transform $u = u(\xi)$, $\xi = kx + \omega t$, equation (68) is transformed into the following ordinary differential equation, i.e:

$$\omega u' + ku u' - \alpha k^2 u'' + \beta k^3 u''' = 0 \quad (69)$$

By integrating the two ends of the equation and making the integration constant zero, then the above equation can be further reduced to:

$$\omega u + \frac{1}{2}ku^2 - \alpha k^2 u' + \beta k^3 u'' = 0 \quad (70)$$

Let equation (69) have a solution of the form $u(\xi) = \sum_{i=0}^p a_i e^{i\xi} / \sum_{j=0}^q b_j e^{j\xi}$. By balancing the nonlinear term and the highest order derivative term, we can obtain $p = q$ in the solution, whereby making $p = q = 2$, the solution of equation (70) can be expressed as

$$u(\xi) = \frac{a_0 + a_1 e^\xi + a_2 e^{2\xi}}{b_0 + b_1 e^\xi + b_2 e^{2\xi}} \quad (71)$$

Substituting equation (71) into equation (70), combining all terms of e^ξ with the same power and setting the coefficient of $e^{i\xi}$ to zero gives the following system of algebraic equations, i.e:

$$\begin{aligned} e^{0\xi} : & 2\omega a_0 b_0^2 + ka_0^2 b_0 = 0 \\ e^\xi : & 2\beta k^3 a_1 b_0^2 + 2\omega a_1 b_0^2 + 2ka_0 a_1 b_0 - 2\beta k^3 a_0 b_1 b_0 - 2\alpha k^2 a_1 b_0^2 \\ & + ka_0 b_1 + 4\omega a_0 b_1 b_0 + 2\alpha k^2 a_0 b_1 b_0 = 0 \\ e^{2\xi} : & 2\omega a_0 b_1^2 + 2\beta k^3 a_0 b_1^2 + ka_0^2 b_2 + ka_1^2 b_0 + 4\omega a_0 b_2 b_0 + 2ka_0 a_2 b_0 \\ & + 2\alpha k^2 a_0 b_1^2 + 4\omega a_1 b_0 b_1 - 2\beta k^3 a_1 b_0 b_1 + 2ka_0 a_1 b_1 + 2\omega a_2 b_0^2 \\ & + 8\beta k^3 a_2 b_0^2 + 4\alpha k^2 a_0 b_2 b_0 - 8\beta k^3 a_0 b_2 b_0 - 4\alpha k^2 a_2 b_0^2 \\ & - 2\alpha k^2 a_1 b_0 b_1 = 0 \\ e^{3\xi} : & 6\beta k^3 a_2 b_0 b_1 + 6b_1 \alpha k^2 a_0 b_2 - 12\beta k^3 a_1 b_0 b_2 + 2ka_1 a_2 b_0 + 6b_1 \beta k^3 a_0 b_2 \\ & + 4\omega a_1 b_0 b_2 + ka_1^2 b_1 + 4\omega a_2 b_0 b_1 - 6k^2 a_2 b_0 b_1 + 2\omega a_1 b_1^2 \\ & + 2ka_0 a_1 b_2 + 2ka_0 a_2 b_1 + 4b_1 \omega a_0 b_2 = 0 \\ e^{4\xi} : & 4\alpha k^2 a_0 b_2^2 + 2\omega a_0 b_2^2 + 4\omega a_2 b_0 b_2 + 2ka_1 a_2 b_1 - 2\alpha k^2 a_2 b_1^2 + 2\beta k^3 a_2 b_1^2 \\ & + 2\alpha k^2 a_1 b_1 b_2 - 4\alpha k^2 a_2 b_0 b_2 - 2\beta k^3 a_1 b_1 b_2 + 2\omega a_2 b_1^2 + ka_2^2 b_0 \\ & + 8\beta k^3 a_0 b_2^2 + 2ka_0 a_2 b_2 + 4\omega a_1 b_1 b_2 - 8\beta k^3 a_2 b_0 b_2 + ka_1^2 b_2 = 0 \\ e^{5\xi} : & 2\alpha k^2 a_1 b_2^2 + ka_2^2 b_1 + 4\omega a_2 b_1 b_2 + 2\beta k^3 a_1 b_2^2 + 2\omega a_1 b_2^2 \\ & - 2\alpha k^2 a_2 b_1 b_2 + 2ka_1 a_2 b_2 - 2\beta k^3 a_2 b_1 b_2 = 0 \\ e^{6\xi} : & 2\omega a_2 b_2^2 + ka_2^2 b_2 = 0 \end{aligned} \quad (72)$$

By solving equation (72) with the help of Maple, the following four sets of parameter values are obtained. First set of parameters:

$$k = -\frac{\alpha}{5\beta}, \quad \omega = -\frac{6\alpha^3}{125\beta^2}, \quad a_0 = -\frac{3\alpha^2 b_1^2}{25\beta b_2}, \quad a_1 = 0, \quad a_2 = 0, \quad b_0 = \frac{b_1^2}{4b_2} \quad (73)$$

Second set of parameters:

$$k = -\frac{\alpha}{5\beta}, \omega = \frac{6\alpha^3}{125\beta^2}, a_0 = 0, a_1 = \frac{12\alpha^2b_1}{25\beta}, a_2 = \frac{12\alpha^2b_2}{\beta}, b_0 = \frac{b_1^2}{4b_2} \quad (74)$$

Third set of parameters:

$$k = \frac{\alpha}{5\beta}, \omega = -\frac{6\alpha^3}{125\beta^2}, a_0 = \frac{12\alpha^2b_1^2}{25\beta b_2}, a_1 = \frac{12\alpha^2b_1}{\beta}, a_2 = 0, b_0 = \frac{b_1^2}{4b_2} \quad (75)$$

Fourth set of parameters:

$$k = \frac{\alpha}{5\beta}, \omega = \frac{6\alpha^3}{125\beta^2}, a_0 = 0, a_1 = 0, a_2 = -\frac{12\alpha^2b_2}{25\beta}, b_0 = \frac{b_1^2}{4b_2} \quad (76)$$

Now take the first set of parameters and find one of the exact sets of solutions of the KdV-Burgers equation.

Substituting the values of the first set of parameters into equation (71), combined with $u = u(\xi)$, $\xi = kx + \omega t$, yields one of the exact sets of solutions to the KdV-Burgers equation as:

$$u(x, t) = -\frac{12}{25} \frac{\alpha^2 b_1^2}{\beta} \cdot \frac{1}{b_1^2 + 4b_1 e^{-\frac{1}{125} \frac{\alpha(25x\beta + 6\alpha^2 t)}{\beta^2}} b_2 + 4b_2^2 e^{-\frac{2}{125} \frac{\alpha(25x\beta + 6\alpha^2 t)}{\beta^2}}} \quad (77)$$

For Eq. (77), if the constant in Eq. is taken to be $b_1 = 1, b_2 = 1, \omega = 0.5, k = 1, \alpha = 1.5, \beta = 2.5$, where x, t takes on a range of values of $x \in [-40, \dots, 90], t \in [0, \dots, 5]$, the three-dimensional diagram of its exact solution is shown in Fig. 6.

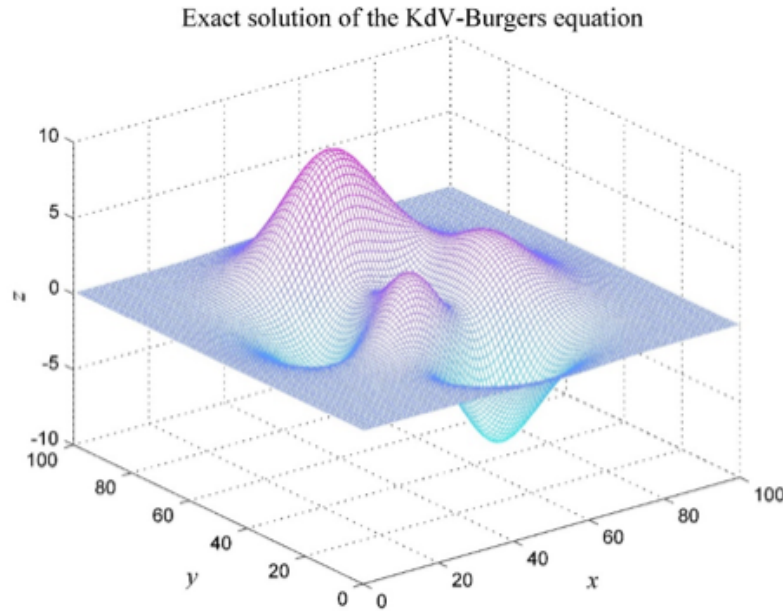


Figure 6: Exact solution of the KdV-Burgers equation

Similarly, the exact solution to the KdV-Burgers equation can be obtained for the remaining three sets of parameters by the process described above and will not be repeated here.

5. Conclusion

As non-linear mathematical physics equations describe practical problems in a number of engineering fields, including physics and fluid dynamics, it is essential to solving them exactly. In this paper, the exact solutions of the rod equation, the canonical long-wave equation, and the KdV-Burgers equation are solved using a geometric grid factorisation algorithm in terms of isolator theory and symbolic computing systems, and the following conclusions are drawn:

1. For the rod equation, a periodic wave becomes an anti-clock-type isolated wave when $c < 0$, $g_1(c) < g < g_2(c)$, φ_0 satisfies $\varphi_-^* < \varphi_0 < \bar{\varphi}_+$ and φ_0 converges to φ_-^* . This also shows that it is possible to cross the odd line of a system of rod equations in one of its orbits by means of a geometric grid factorisation, thus also demonstrating the superiority and validity of using this algorithm for the study of isolated waves.
2. For the regular long-wave equation and the KdV-Burgers equation, solutions of the form $u(\xi) = \sum_{i=0}^p a_i e^{i\xi} / \sum_{j=0}^q b_j e^{i\xi}$ can be efficiently decomposed into different parametric forms by the geometric grid factorisation algorithm, and different exact solutions can be obtained by solving for different parameters.

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