

Deferred Order Convergences

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Abstract: In this study, we introduce the concepts of deferred order convergence and deferred order statistical convergence and investigate the relations between these concepts.

Keywords: Statistical convergence, order convergence, deferred Cesàro mean, deferred convergence, deferred order statistical convergence

1 Introduction

A sequence $x = (x_k)$ is called statistically convergent to ℓ if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0,$$

where the vertical bars denote cardinality of the set.

Statistical convergence was introduced by Steinhaus in [1], Fast in [2] and also by Buck in [3] independently and since then has been studied by other authors including [4–8].

A lacunary sequence [9] is an increasing integer sequence $\theta = \{k_n\}$ such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \rightarrow \infty$ as $n \rightarrow \infty$. The intervals determined by θ is denoted by $I_n = (k_{n-1}, k_n]$.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ , and $\lambda_{n+1} - \lambda_n \leq 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean [10] is defined by

$$\frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^n x_k.$$

If

$$(a) \quad \lim_{n \rightarrow \infty} \frac{1}{h_n} \sum_{k \in I_n} |x_k - \ell| = 0,$$

$$(b) \quad \lim_{n \rightarrow \infty} \frac{1}{h_n} |\{k \in I_n : |x_k - \ell| \geq \varepsilon\}| = 0,$$

(c)

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^n |x_k - \ell| = 0,$$

(d)

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0,$$

then the sequence $x = (x_k)$ is said to be strongly lacunary convergent or $[N_\theta]$ -convergent, lacunary statistically convergent or S_θ -convergent, strongly λ -convergent and λ -statistically convergent to ℓ respectively. λ -statistically convergent sequence was defined by Mursaleen in [11] and lacunary statistically convergent or S_θ -convergent sequence was defined in [6] by Fridy and Orhan.

Let $B = (b_{nk})$ be an infinite matrix and $x = (x_k)$ be a sequence of complex numbers. The sequence $B_n(x)$ defined by

$$B_n(x) = \sum_{k=1}^{\infty} a_{kn} x_k$$

is called the B -transform of x whenever the series converges each $n = 1, 2, \dots$. The sequence $x = (x_k)$ is said to be B -summable to ℓ if $\lim_{n \rightarrow \infty} B_n(x) = \ell$. A matrix B is said to be regular if B transforms every convergent sequence to convergent sequence by preserving the limit. Silverman-Toeplitz Theorem states necessary and sufficient conditions for regularity of $B = (b_{nk})$ as follows:

1.

$$\sup_n \sum_{k=1}^{\infty} |b_{nk}| < \infty$$

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2.

$$\lim_{n \rightarrow \infty} b_{nk} = 0 \quad \text{for each } k \in \mathbb{N}$$

3.

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk} = 1.$$

For definitions of partially ordered set, lattice, metric additive system (mas) see [12–16].

Let L be an additive system and Γ be a real valued function on L . Then a function γ is defined on L by

$$\gamma(a, b) = \Gamma(a \vee b) - \Gamma(a) - \Gamma(b).$$

Definition 1. Let (x_n) be in a mas (L, γ) . If there exists a sequence (y_n) in L with $y_n \downarrow \theta$ such that

$$\gamma(x_n, x) < \Gamma(y_n)$$

for all $n \in \mathbb{N}$, then (x_n) is said to be order convergent to $x \in L$.

Definition 2. [17] Let (x_n) be in a mas (L, γ) . If there exists a sequence (y_n) in L with $y_n \downarrow \theta$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \delta(x_k, x) \geq \Gamma(y_k)\}| = 0,$$

then (x_n) is said to be order statistically convergent to $x \in L$.

Later, in [18] Biswas and Ganguly studied order ideal convergence in a metric additive system.

2 Deferred Order Cesàro and Deferred Order Statistical Convergence

A sequence $x = (x_k)$ is said to be strongly Cesàro summable to the number ℓ if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |x_k - \ell| = 0.$$

R.P. Agnew [19] defined the deferred Cesàro mean as a generalization of Cesàro mean of real (or complex) valued sequence $x = (x_k)$ by

$$(D_{\alpha, \beta} x)_n := \frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} x_k, \quad n = 1, 2, 3, \dots,$$

where $\alpha = \{\alpha(n) : n \in \mathbb{N}\}$ and $\beta = \{\beta(n) : n \in \mathbb{N}\}$ are the sequences of nonnegative integers satisfying $\alpha(n) < \beta(n)$ and $\lim_{n \rightarrow \infty} \beta(n) = \infty$.

A sequence $x = (x_k)$ is said to be strongly $D_{\alpha, \beta}$ -convergent to $\ell \in \mathbb{R}$ if

$$\lim_{n \rightarrow \infty} \frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} |x_k - \ell| = 0$$

Strongly $D_{\alpha, \beta}$ -convergent sequence $x = (x_k)$ is strongly Cesàro convergent if and only if

$$\sup_n \frac{\alpha(n)}{\beta(n) - \alpha(n)} < \infty.$$

Definition 3. A sequence (x_n) in a metric additive system (L, γ) is said to be order Cesàro convergent (i.e. OC-convergent) to $x \in L$ if, there exists a sequence (y_n) in L with $y_n \downarrow \theta$ such that

$$\frac{1}{n} \sum_{k=1}^n \gamma(x_k, x) < \Gamma(y_n)$$

for all $n \in \mathbb{N}$.

The concept of deferred statistical convergent sequence was introduced in [20] as follows:

If for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\alpha(n) - \beta(n)} |\{\alpha(n) < k \leq \beta(n) : |x_k - \ell| \geq \varepsilon\}| = 0$$

then the sequence $x = (x_k)$ is said to be deferred statistically convergent to the number ℓ .

Definition 4. A sequence (x_n) in a metric additive system (L, γ) is said to be deferred order Cesàro convergent (i.e. DOC-convergent) to $x \in L$ if, there exists a sequence (y_n) in L with $y_n \downarrow \theta$ and a number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} \gamma(x_k, x) < \Gamma(y_n)$$

for all $n \in \mathbb{N}$.

It is clear that;

- (i) If $\beta(n) = n$ and $\alpha(n) = 0$, then Definition 4 coincides with the definition of order Cesàro convergence,
- (ii) Let $\theta = (k_n)$ be a lacunary sequence. If we consider $\beta(n) = k_n$ and $\alpha(n) = k_{n-1}$, then Definition 4 coincides with the order lacunary convergence,
- (iii) If $\beta(n) = n$ and $\alpha(n) = n - \lambda_n$, then Definition 4 coincides with the order λ -convergence.

Definition 5. A sequence (x_n) in a metric additive system (L, γ) is said to be deferred order statistically convergent (i.e. DOS-convergent) to $x \in L$ if, there exists a sequence (y_n) in L with $y_n \downarrow \theta$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\beta(n) - \alpha(n)} |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| = 0.$$

It is clear that;

- (i) If $\beta(n) = n$ and $\alpha(n) = 0$, then Definition 5 coincides with the definition of order statistical convergence,
- (ii) Let $\theta = (k_n)$ be a lacunary sequence. If we consider $\beta(n) = k_n$ and $\alpha(n) = k_{n-1}$, then Definition 5 coincides with the order lacunary statistical convergence,
- (iii) If $\beta(n) = n$ and $\alpha(n) = n - \lambda_n$, then Definition 5 coincides with the order λ -statistical convergence of sequences.

3 Inclusion Relations

Theorem 1. Let $\{\alpha(n)\}, \{\beta(n)\}, \{\alpha'(n)\}$ and $\{\beta'(n)\}$ be sequences of non-negative integers satisfying $\alpha(n) \leq \alpha'(n) < \beta'(n) \leq \beta(n)$ for all $n \in \mathbb{N}$ and

$$\limsup_{n \rightarrow \infty} \frac{\beta(n) - \alpha(n)}{\beta'(n) - \alpha'(n)} < \infty,$$

then $x_n \rightarrow x(DOS[\alpha, \beta])$ implies $x_n \rightarrow x(DOS[\alpha', \beta'])$.

Proof. From the following inclusion

$$\{\alpha'(n) < k \leq \beta'(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}$$

$$\subset \{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}$$

we can write

$$\begin{aligned} & \frac{1}{\beta'(n) - \alpha'(n)} \\ & \times |\{\alpha'(n) < k \leq \beta'(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| \\ & \leq \frac{\beta(n) - \alpha(n)}{\beta'(n) - \alpha'(n)} \frac{1}{\beta(n) - \alpha(n)} \\ & \times |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}|. \end{aligned}$$

After taking limit when $n \rightarrow \infty$ desired result is obtained.

Theorem 2. Let $\{\alpha(n)\}, \{\beta(n)\}, \{\alpha'(n)\}$ and $\{\beta'(n)\}$ be sequences of non-negative integers satisfying $\alpha(n) \leq \alpha'(n) < \beta'(n) \leq \beta(n)$ for all $n \in \mathbb{N}$ and

$$\limsup_{n \rightarrow \infty} \frac{\beta(n) - \alpha(n)}{\beta'(n) - \alpha'(n)} < \infty,$$

then $x_n \rightarrow x(DOC[\alpha, \beta])$ implies $x_n \rightarrow x(DOC[\alpha', \beta'])$.

Since the proof is similar to the proof of Theorem 1, we omit it.

Theorem 3. Let D be a bounded function on L . If (x_n) is deferred order Cesàro convergent to $x \in L$, then (x_n) is deferred order statistically convergent to x , that is, if $x_n \rightarrow x(DOC[\alpha, \beta])$, then $x_n \rightarrow x(DOS[\alpha, \beta])$.

Proof. Let $x_n \rightarrow x(DOC[\alpha, \beta])$. Since $y_n \downarrow \theta$, then $\Gamma(y_n) \downarrow 0$ and consequently $(\Gamma(y_n))$ is bounded. Say $m = \inf_k \Gamma(y_k)$.

For (y_n) in L with $y_n \downarrow \theta$, we get

$$\begin{aligned} \Gamma(y_n) & \geq \frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} \gamma(x_k, x) \\ & = \frac{1}{\beta(n) - \alpha(n)} \left(\sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) \geq \Gamma(y_k)}}^{\beta(n)} + \sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) < \Gamma(y_k)}}^{\beta(n)} \right) \gamma(x_k, x) \\ & \geq \frac{1}{\beta(n) - \alpha(n)} \sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) \geq \Gamma(y_k)}}^{\beta(n)} \gamma(x_k, x) \\ & \geq \frac{1}{\beta(n) - \alpha(n)} \sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) \geq \Gamma(y_k)}}^{\beta(n)} \Gamma(y_k) \\ & \geq m \frac{1}{\beta(n) - \alpha(n)} \\ & \times |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}|. \end{aligned}$$

Hence, by above inequality and since $\lim_{n \rightarrow \infty} \Gamma(y_n) = \Gamma(\theta) = 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\beta(n) - \alpha(n)} \times |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| = 0,$$

that is $x_n \rightarrow x(DOS[\alpha, \beta])$.

Theorem 4. Let $\frac{\alpha(n)}{\beta(n) - \alpha(n)}$ be bounded. If (x_n) is order bounded and deferred order statistically convergent to x , then (x_n) is deferred order Cesàro convergent to x , that is, if x order bounded and $x_n \rightarrow x(DOS[\alpha, \beta])$, then $x_n \rightarrow x(DOC[\alpha, \beta])$.

Proof. Suppose that $x_n \rightarrow x(DOS[\alpha, \beta])$ and (x_n) is order bounded, say $\gamma(x_k, x) \leq K$ for all k . Since $\Gamma(y_n) \rightarrow 0$ as $y_n \downarrow \theta$, $(\Gamma(y_n))$ is Cesàro convergent to 0, since $\frac{\alpha(n)}{\beta(n) - \alpha(n)}$ is bounded $(\Gamma(y_n))$ is deferred Cesàro convergent to 0 so, we can write

$$\lim_{n \rightarrow \infty} \frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} \Gamma(y_k) = 0$$

and

$$\begin{aligned}
 & \frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} \gamma(x_k, x) \\
 &= \frac{1}{\beta(n) - \alpha(n)} \left(\sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) \geq \Gamma(y_k)}}^{\beta(n)} + \sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) < \Gamma(y_k)}}^{\beta(n)} \right) \gamma(x_k, x) \\
 &\leq \frac{1}{\beta(n) - \alpha(n)} \\
 &\times \left(K \sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) \geq \Gamma(y_k)}}^{\beta(n)} 1 + \sum_{\substack{k=\alpha(n)+1 \\ \gamma(x_k, x) < \Gamma(y_k)}}^{\beta(n)} \gamma(x_k, x) \right) \\
 &\leq K \frac{1}{\beta(n) - \alpha(n)} \\
 &\times |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| \\
 &+ \frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} \Gamma(y_k)
 \end{aligned}$$

hence we have,

$$\frac{1}{\beta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\beta(n)} \gamma(x_k, x) < \Gamma(y_n)$$

for all $n \in \mathbb{N}$.

Theorem 5. If the sequence $\left\{ \frac{\alpha(n)}{\beta(n) - \alpha(n)} \right\}_{n \in \mathbb{N}}$ is bounded, then $x_n \rightarrow x(OS)$ implies $x_n \rightarrow L(DOS[\alpha, \beta])$.

Proof. Let $x_n \rightarrow x(OS)$ then there exists a sequence (y_n) in L with $y_n \downarrow \theta$ such that,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \gamma(x_k, x) \geq \Gamma(y_k)\}| = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{\beta(n)} |\{k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| = 0.$$

From the following inclusion

$$\begin{aligned}
 & \{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\} \\
 & \subseteq \{k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}
 \end{aligned}$$

and the inequality

$$\begin{aligned}
 & |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| \\
 & \leq |\{k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}|
 \end{aligned}$$

we have

$$\begin{aligned}
 & \frac{1}{\beta(n) - \alpha(n)} \\
 & \times |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| \\
 &= \frac{\beta(n) - \alpha(n) + \alpha(n)}{\beta(n) - \alpha(n)} \frac{1}{\beta(n)} \\
 & \times |\{\alpha(n) < k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}| \\
 &\leq \left(1 + \frac{\alpha(n)}{\beta(n) - \alpha(n)} \right) \frac{1}{\beta(n)} \\
 & \times |\{k \leq \beta(n) : \gamma(x_k, x) \geq \Gamma(y_k)\}|
 \end{aligned}$$

and we obtain $x_n \rightarrow x(DOS[\alpha, \beta])$.

Theorem 6. Let $\beta(n) = n$ for all $n \in \mathbb{N}$. Then, $x_n \rightarrow x(DOS[\alpha, n])$ if and only if $x_n \rightarrow x(OS)$.

Proof. Assume that $x_n \rightarrow x(DOS[\alpha, n])$. By using the technique which was used by Agnew in [19], for each $n \in \mathbb{N}$, letting $\alpha(n) = n^{(1)}$, $\alpha(n^{(1)}) = n^{(2)}$, $\alpha(n^{(2)}) = n^{(3)}, \dots$, we may write

$$\begin{aligned}
 & \{k \leq n : \gamma(x_k, x) \geq \Gamma(y_k)\} = \\
 & \{k \leq n^{(1)} : \gamma(x_k, x) \geq \Gamma(y_k)\} \\
 & \cup \{n^{(1)} < k \leq n : \gamma(x_k, x) \geq \Gamma(y_k)\}, \\
 & \{k \leq n^{(1)} : \gamma(x_k, x) \geq \Gamma(y_k)\} = \\
 & \{k \leq n^{(2)} : \gamma(x_k, x) \geq \Gamma(y_k)\} \\
 & \cup \{n^{(2)} < k \leq n^{(1)} : \gamma(x_k, x) \geq \Gamma(y_k)\} \\
 & \text{and} \\
 & \{k \leq n^{(2)} : \gamma(x_k, x) \geq \Gamma(y_k)\} = \\
 & \{k \leq n^{(3)} : \gamma(x_k, x) \geq \Gamma(y_k)\} \\
 & \cup \{n^{(3)} < k \leq n^{(2)} : \gamma(x_k, x) \geq \Gamma(y_k)\}.
 \end{aligned}$$

This process may be continued until for some positive integer h depending on n , we obtain

$$\begin{aligned}
 & \{k \leq n^{(h-1)} : \gamma(x_k, x) \geq \Gamma(y_k)\} = \\
 & \{k \leq n^{(h)} : \gamma(x_k, x) \geq \Gamma(y_k)\} \\
 & \cup \{n^{(h)} < k \leq n^{(h-1)} : \gamma(x_k, x) \geq \Gamma(y_k)\}
 \end{aligned}$$

for each m where $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$. Therefore, we can write

$$\frac{1}{n} |\{k \leq n : \gamma(x_k, x) \geq \Gamma(y_k)\}| = \sum_{r=0}^h \frac{n^{(r)} - n^{(r+1)}}{n} d_r$$

for every n , where

$$d_r = \frac{1}{n^{(r)} - n^{(r+1)}}$$

$$\times |\{n^{(r+1)} < k \leq n^{(r)} : \gamma(x_k, x) \geq \Gamma(y_k)\}|.$$

If we consider a matrix $B = (b_{nr})$ as

$$b_{nr} = \begin{cases} \frac{n^{(r)} - n^{(r+1)}}{n}, & r = 0, 1, 2, \dots, h \\ 0, & \text{otherwise} \end{cases}$$

where $n^{(0)} = n$, then the sequence

$$\left\{ \frac{1}{n} |\{k \leq n : \gamma(x_k, x) \geq \Gamma(y_k)\}| \right\}$$

is the (b_{nr}) transformation of the sequence (d_r) . Since $n^{(r)} > n^{(r+1)}$, $r = 1, 2, \dots, h$, and $n^{(h+1)} = 0$, this transformation evidently satisfies (1) and (3); and for fixed k , $\frac{n^{(k)} - n^{(k+1)}}{n}$ is either zero or a fraction of which the denominator is n and the numerator is $\leq k$ so that (2) holds. Therefore, the matrix (b_{nr}) is a regular and since the sequence

$$\left\{ \frac{1}{n^{(r)} - n^{(r+1)}} |\{n^{(r+1)} < k \leq n^{(r)} : \gamma(x_k, x) \geq \Gamma(y_k)\}| \right\}$$

is convergent to zero, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \gamma(x_k, x) \geq \Gamma(y_k)\}| = 0.$$

Conversely, since $\left\{ \frac{\alpha(n)}{\beta(n) - \alpha(n)} \right\}_{n \in \mathbb{N}}$ is bounded for $\beta(n) = n$, by Theorem 5, we have $x_n \rightarrow x(OS)$ implies $x_n \rightarrow x(DOS[\alpha, \beta])$.

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