

A Note About The Convolved Fibonacci Polynomial Sequences

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Abstract: After recalling the convolved Fibonacci polynomials considered in literature, a particular set of convolved Fibonacci polynomials is introduced in order to find explicit expressions for product sums of Fibonacci numbers, in terms of the Golden ratio. Recurrence relations are determined by using a preceding more general result.

Keywords: Fibonacci numbers, convolved Fibonacci numbers and polynomials, recurrence relations

1 Introduction

Convolved Fibonacci numbers and polynomials have been considered in past time [1, 2, 4], and even recently [3, 5, 6, 8–10] in connection with the problem of finding representation formulas for product sums of Fibonacci numbers, and related subjects, including relations with the second kind Chebyshev polynomials.

It seems that the subject is of some interest, since several authors have been attracted by it, mainly in connection with problems of Number theory, for some references see, e.g., Sloane [12].

In this article, for the convolved Fibonacci polynomial sequence already considered in literature, we recover the explicit form and derive the recurrence relation, by using a general result stated in [11].

Furthermore, in Section 4, we introduce a new convolved Fibonacci polynomial sequence which still gives back, at the point $x = 1$, the Fibonacci numbers, but using polynomials of lower degrees and lower coefficients with respect to the preceding one. For this convolved polynomials we determine the recurrence relation and their explicit form, in which the Golden ratio Φ appears. Representation formulas of sums of products of Fibonacci numbers in terms of the Golden ratio are also shown (Corollaries 4.3 and 4.4). Lastly, we show their further connections with the Fibonacci numbers and we derive multiplication formulas by using convolution (Theorem 4.5).

2 Definitions

We start from the sequence of Fibonacci numbers defined by the recursion:

$$\begin{aligned} f_0 &= 1, f_1 = 1, \\ f_{n+1} &= f_n + f_{n-1}, \quad (n \geq 1). \end{aligned} \quad (1)$$

so that its generating function is given by:

$$\frac{1}{1-t-t^2} = \sum_{n=0}^{\infty} f_n t^n. \quad (2)$$

The Fibonacci polynomials are defined by the recursion:

$$\begin{aligned} F_0(x) &= 1, F_1(x) = 1, \\ F_{n+1}(x) &= xF_n(x) + F_{n-1}(x), \quad (n \geq 1), \end{aligned} \quad (3)$$

and are also given by the generating function:

$$\frac{1}{1-xt-t^2} = \sum_{n=0}^{\infty} F_n(x) t^n, \quad (4)$$

therefore, it results $F_n(1) = f_n$.

These polynomials are very important, as they frequently are used in Number theory and in applied mathematics. Many scholars studied the properties of the sequences $\{f_n\}$ and $\{F_n(x)\}$ and obtained important results. For

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example, Yi and Zhang [4] obtained representation formulas for the summation of products of Fibonacci polynomials (in particular numbers), and T. Kim et al. [5] introduced a new approach to the convolved Fibonacci numbers. They used the relevant generating function in order to find new explicit identities for the convolved Fibonacci numbers.

Actually other sequences related to the Fibonacci polynomials and numbers have been considered in literature (see e.g. [1, 2, 9]), and also recently studied [3, 10], in particular the so-called convolved Fibonacci polynomial sequences, which will be considered in what follows.

The characteristic equation associated with the recursion (3) is:

$$t^2 = xt + 1 \quad (5)$$

and their roots are:

$$t_1 = \frac{x + \sqrt{x^2 + 4}}{2}, \quad t_2 = \frac{x - \sqrt{x^2 + 4}}{2}. \quad (6)$$

The first few Fibonacci polynomials $F_n(x)$ are:

$$\begin{aligned} F_0(x) &= 1 \\ F_1(x) &= x \\ F_2(x) &= x^2 + 1 \\ F_3(x) &= x^3 + 2x \\ F_4(x) &= x^4 + 3x^2 + 1 \\ F_5(x) &= x^5 + 4x^3 + 3x \end{aligned}$$

Further values can be found by recursion.

3 The convolved Fibonacci polynomial sequence

The convolved Fibonacci polynomials are defined by the generating function:

$$\left(\frac{1}{1 - xt - t^2} \right)^x = \sum_{n=0}^{\infty} \mathcal{F}_n(x) t^n. \quad (7)$$

The roots of the equation $t^2 + xt - 1 = 0$ are:

$$t_1 = \frac{-x - \sqrt{x^2 + 4}}{2}, \quad \text{and} \quad t_2 = \frac{-x + \sqrt{x^2 + 4}}{2},$$

$$(t_1 t_2 = -1).$$

Recalling the binomial expansion:

$$\frac{1}{(1-t)^x} = \sum_{n=0}^{\infty} \frac{x^{(n)}}{n!} t^n, \quad (8)$$

where $x^{(0)} := 1$, $x^{(n)} := x(x+1) \cdots (x+n-1)$, and noting that

$$\begin{aligned} \left(\frac{1}{1 - xt - t^2} \right)^x &= \left(\frac{1}{(t_1 - t)(t - t_2)} \right)^x = \\ &= \frac{1}{\left(1 - \frac{t}{t_1}\right)^x \left(1 - \frac{t}{t_2}\right)^x}, \end{aligned}$$

we find:

$$\begin{aligned} \frac{1}{\left(1 - \frac{t}{t_1}\right)^x \left(1 - \frac{t}{t_2}\right)^x} &= \\ \sum_{n=0}^{\infty} \frac{x^{(n)}}{n!} \left(\frac{t}{t_1} \right)^n \sum_{n=0}^{\infty} \frac{x^{(n)}}{n!} \left(\frac{t}{t_2} \right)^n &= \\ = \sum_{n=0}^{\infty} \sum_{k_1+k_2=n} \left[\frac{x^{(k_1)} x^{(k_2)}}{k_1! k_2!} \frac{1}{t_1^{k_1} t_2^{k_2}} \right] t^n. \end{aligned} \quad (9)$$

Therefore

$$\mathcal{F}_n(x) = \sum_{k_1+k_2=n} \frac{x^{(k_1)} x^{(k_2)}}{k_1! k_2!} \frac{1}{t_1^{k_1} t_2^{k_2}}. \quad (10)$$

Note that both the sequences $F_n(x)$ and $\mathcal{F}_n(x)$ satisfy the condition:

$$F_n(1) = \mathcal{F}_n(1) = f_n. \quad (11)$$

The first few convolved $\mathcal{F}_n(x)$ are:

$$\begin{aligned} \mathcal{F}_0(x) &= 1 \\ \mathcal{F}_1(x) &= x^2 \\ \mathcal{F}_2(x) &= \frac{1}{2!} x(x^3 + x^2 + 2) \\ \mathcal{F}_3(x) &= \frac{1}{3!} x^2(x^4 + 3x^3 + 2x^2 + 6x + 6) \\ \mathcal{F}_4(x) &= \frac{1}{4!} x(x^7 + 6x^6 + 11x^5 + 18x^4 + 36x^3 \\ &\quad + 24x^2 + 12x + 12) \\ \mathcal{F}_5(x) &= \frac{1}{5!} x^2(x^8 + 10x^7 + 35x^6 + 70x^5 + 144x^4 \\ &\quad + 220x^3 + 180x^2 + 180x + 120) \end{aligned}$$

Further values can be found by using a computer algebra program.

In [11], starting from a paper by Chen Zhuoyu and Qi Lan [3], the linear recurrence relations associated with a general sequences of polynomials have been derived. In particular we have:

Theorem 1 The polynomial sequence $\{g_k(x)\}_{k \in \mathbb{N}} \equiv \{g_k(x; a, b)\}_{k \in \mathbb{N}}$ of polynomials defined by the generating function:

$$G(t, x) = \left(\frac{1}{1 + at + bt^2} \right)^x = \sum_{n=0}^{\infty} g_n(x; a, b) \frac{t^n}{n!} \quad (12)$$

satisfy the linear recurrence relation:

$$\begin{aligned} g_k(x) + a(x+k-1)g_{k-1}(x) \\ + b(k-1)(k+2x-2)g_{k-2}(x) = 0. \end{aligned} \quad (13)$$

Putting $a = -x$, $b = -1$ and $g_n(x) := g_n(x; -x, -1)$ we find:

$$\left(\frac{1}{1 - xt - t^2} \right)^x = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}, \quad (14)$$

so that

$$g_n(x) = n! \mathcal{F}_n(x), \quad (15)$$

and the recurrence relation writes:

$$\begin{aligned} n \mathcal{F}_n(x) - x(x+n-1) \mathcal{F}_{n-1}(x) \\ - (2x+n-2) \mathcal{F}_{n-2}(x) = 0. \end{aligned} \quad (16)$$

4 A new convolved polynomial sequence

In what follows, we consider a different convolved polynomial sequence with the purpose to recover the Fibonacci numbers. Theoretically we could use a generating function of the type:

$$\left(\frac{a + (1-a)x}{1 - x^\alpha t - x^\beta t^2} \right)^x, \quad (17)$$

but, of course, we would like to construct the generated polynomials in such a way that they have lower grades, and smaller coefficients, with respect to those considered above.

Then, assuming $a = 1$, $\alpha = 1$ and $\beta = 2$ in (17), we introduce the following convolved Fibonacci sequence, defined by the generating function:

$$\left(\frac{1}{1 - xt - x^2 t^2} \right)^x = \sum_{n=0}^{\infty} \mathbf{F}_n(x) t^n. \quad (18)$$

The roots of the equation $x^2 t^2 + xt - 1 = 0$ are:

$$\begin{aligned} t_1 &= \frac{-(1+\sqrt{5})}{2x} = \frac{-\Phi}{x}, \\ t_2 &= \frac{-(1-\sqrt{5})}{2x} = \frac{\Phi-1}{x}, \\ (t_1 t_2 &= -\frac{1}{x^2}), \end{aligned}$$

where Φ is the Golden ratio.

Recalling the binomial expansion (8), and noting that:

$$\left(\frac{1}{1 - xt - x^2 t^2} \right)^x = \frac{1}{(1 - \frac{t}{t_1})^x (1 - \frac{t}{t_2})^x},$$

we find:

$$\begin{aligned} \frac{1}{(1 - \frac{t}{t_1})^x (1 - \frac{t}{t_2})^x} &= \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{(n)}}{n!} \left(\frac{tx}{\Phi} \right)^n \sum_{n=0}^{\infty} \frac{x^{(n)}}{n!} \left(\frac{tx}{\Phi-1} \right)^n = \\ &= \sum_{n=0}^{\infty} \sum_{k_1+k_2=n} (-1)^{k_1} \frac{x^{(k_1)} x^{(k_2)}}{k_1! k_2!} \frac{x^n}{\Phi^{k_1} (\Phi-1)^{k_2}} t^n. \end{aligned} \quad (19)$$

Therefore

$$\mathbf{F}_n(x) = x^n \sum_{k_1+k_2=n} (-1)^{k_1} \frac{x^{(k_1)} x^{(k_2)}}{k_1! k_2! \Phi^{k_1} (\Phi-1)^{k_2}}. \quad (20)$$

The first few convolved $\mathbf{F}_n(x)$ are:

$$\mathbf{F}_0(x) = 1$$

$$\mathbf{F}_1(x) = x^2$$

$$\mathbf{F}_2(x) = \frac{1}{2!} x^3 (x+3)$$

$$\mathbf{F}_3(x) = \frac{1}{3!} x^4 (x^2 + 9x + 8)$$

$$\mathbf{F}_4(x) = \frac{1}{4!} x^5 (x^3 + 18x^2 + 59x + 42)$$

$$\mathbf{F}_5(x) = \frac{1}{5!} x^6 (x^4 + 30x^3 + 215x^2 + 450x + 264)$$

Further values can be found by using a computer algebra program.

For the convolved Fibonacci polynomial sequence $\mathbf{F}_n(x)$ still holds that $\mathbf{F}_n(1) = f_n$, but, according to the preceding tables, it results that the new sequence realizes this condition with polynomials of lower degree and with smaller coefficients.

By using Theorem 2.1, assuming $a = -x$ and $b = -x^2$, and noting that $g_n(x) = n! \mathbf{F}_n(x)$ we have the result:

Theorem 2 The polynomials of the sequence $\{\mathbf{F}_n(x)\}_{n \in \mathbb{N}}$, defined by the generating function (12), verify the recurrence relation:

$$\begin{aligned} n \mathbf{F}_n(x) - x(x+n-1) \mathbf{F}_{n-1}(x) \\ - x^2(2x+n-2) \mathbf{F}_{n-2}(x) = 0. \end{aligned} \quad (21)$$

4.1 Further connections with Fibonacci numbers

We prove the following result

Theorem 3 For any integer n , the numerical sequence $\mathbf{F}_n(2)$ can be expressed, in terms of Fibonacci numbers, by the equation:

$$\mathbf{F}_n(2) = 2^n \sum_{k=0}^n f_k f_{n-k}. \quad (22)$$

and more generally, for any integer $m \geq 2$, it results:

$$\mathbf{F}_n(m) = m^n \sum_{k_1+\dots+k_m=n} f_{k_1} f_{k_2} \dots f_{k_m}. \quad (23)$$

Proof. By definition (18), putting $x = 2$, we find:

$$\left(\frac{1}{1 - 2t - 4t^2} \right)^2 = \sum_{n=0}^{\infty} \mathbf{F}_n(2) t^n, \quad (24)$$

but, recalling equation (2), we can write:

$$\begin{aligned} \left(\frac{1}{1-2t-4t^2} \right)^2 &= \left(\frac{1}{1-(2t)-(2t)^2} \right)^2 = \\ &= \left(\sum_{n=0}^{\infty} f_n 2^n t^n \right)^2 = \sum_{n=0}^{\infty} \left(2^n \sum_{k=0}^n f_k f_{n-k} \right) t^n, \end{aligned} \quad (25)$$

so equation (22) follows.

In general, we find:

$$\left(\frac{1}{1-mt-m^2t^2} \right)^m = \sum_{n=0}^{\infty} F_n(m) t^n, \quad (26)$$

but, recalling equation (2), we can write:

$$\begin{aligned} \left(\frac{1}{1-mt-m^2t^2} \right)^m &= \left(\frac{1}{1-(mt)-(mt)^2} \right)^m = \\ &= \left(\sum_{n=0}^{\infty} f_n m^n t^n \right)^m = \\ &= \sum_{n=0}^{\infty} \left(m^n \sum_{k_1+\dots+k_m=n} f_{k_1} f_{k_2} \dots f_{k_m} \right) t^n, \end{aligned} \quad (27)$$

and equation (23) is proved.

Note that, since $2^{(k_1)} = (k_1 + 1)!$ and $2^{(k_2)} = (k_2 + 1)!$, recalling equation (20), it follows:

$$\begin{aligned} F_n(2) &= 2^n \sum_{k=0}^n f_k f_{n-k} = \\ &= 2^n \sum_{k_1+k_2=n} (-1)^{k_1} \frac{(k_1+1)(k_2+1)}{\Phi^{k_1}(\Phi-1)^{k_2}}. \end{aligned} \quad (28)$$

So, we have the following Corollary of Theorem 4.2, connecting the Fibonacci numbers with the Golden ratio Φ :

Corollary 4 For any integer $n \geq 1$, we have:

$$\sum_{k=0}^n f_k f_{n-k} = \sum_{h=0}^n (-1)^h \frac{(n+1)(n-h+1)}{\Phi^h (\Phi-1)^{n-h}}, \quad (29)$$

and more generally, recalling equations (20) and (23), we find:

Corollary 5 For any integer $n \geq 1$, we have:

$$\begin{aligned} \sum_{k_1+k_2+\dots+k_m=n} f_{k_1} f_{k_2} \dots f_{k_m} &= \\ \sum_{h=0}^n (-1)^h \frac{m^{(h)} m^{(n-h)}}{h! (n-h)! \Phi^h (\Phi-1)^{n-h}}. \end{aligned} \quad (30)$$

4.2 Multiplication formulas by convolution

It is also possible to obtain duplication, and in general multiplication formulas, for the argument of the functions $F_n(x)$.

Theorem 6 For the polynomial sequence $F_n(x)$ the duplication formula holds:

$$F_n(2x) = 2^n \sum_{k=0}^n F_k(x) F_{n-k}(x). \quad (31)$$

and in general, for any integer $m \geq 2$, the multiplication formula holds:

$$F_n(mx) = m^n \sum_{k_1+\dots+k_m=n} F_{k_1}(x) F_{k_2}(x) \dots F_{k_m}(x). \quad (32)$$

Proof. - The result is obtained by the same technique used in Theorem 4.2. Note that Theorem 4.2, assuming $x = 1$, becomes a Corollary of Theorem 4.3.

5 Conclusion

We have shown the recurrence relation for the convolved Fibonacci polynomials, starting from a general result proven in [11]. We have introduced a new type of convolved Fibonacci polynomials, in order to determine explicit formulas for product sums of the Fibonacci numbers. The derived equations connect Fibonacci numbers with the Golden ratio Φ , apparently in a simple and natural form.

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Compliance with ethical standards

Conflict of interest The authors declare that they have not received funds from any institution and that they have no conflict of interest.

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