

Cone Metric Spaces and Cone Two Metric Spaces

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Abstract: There are many generalization forms of metric spaces. Some of them are; fuzzy metric space, cone metric space, K-metric space and K-normed space ... In 2007, Chinese mathematicians Zang and Huang described the cone metric spaces as unaware of the existence of the K-metric and K-normed which was defined and used in the 20th century. At both of them, E Banach space was handled instead of real number. Then, by making further, Huang and Zang have the definition of convergence of series at cone metric spaces [1]. In this presentation, cone metric spaces which are a generalization of metric spaces will be handled. Namely, cone two metric spaces will be examined by using two metric spaces instead of known d- metric space by looking at cone metric spaces. The knowledge about convergence of series will be given by using two metric at cone metric spaces. Some theorems and definitions related to cone two metric spaces will be examined and concluded with examples.

Keywords: Metric spaces, cone metric spaces, cone two metric spaces

1 Introduction

The concept of generalized metric spaces known as D-Metric Spaces was introduced by B.C.Dhage [7]. Huang and Zhang generalized the concept of metric space by replacing the set of real numbers by an ordered Banach space and obtained fixed point theorems for mappings satisfying various conditions [3]. S. Rezapour and R.Hamlbarani showed existence of non-normal cone metric spaces and obtained some fixed point results in cone metric spaces [8].

2 Preliminary Notes

Let E be a real Banach space. A nonempty convex closed subset $P \subseteq E$ is called a **cone** in E if it satisfies:

- i) P is closed, nonempty and $P \neq \{\theta\}$.
- ii) $a, b \in P; a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.
- iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subseteq E$ that is, $x \leq y$ if and only if $-x \in P$. Also we write $x < y$ if $-x \in \text{int}P$.

In the following we always suppose that E is a real Banach space, P is a cone in E and $<<$ is partial ordering

with respect to P [2].

Definition 2.1: A metric space is an ordered pair (X, d) consisting of a set X together with a function $d : X \times X \rightarrow R$ satisfying for $x, y, z \in X$:

- i) $d(x, y) \geq 0$
- ii) $d(x, x) = 0; d(x, y) = 0$ implies $x = y$
- iii) $d(x, y) = d(y, x)$ (symmetry)
- iv) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

The function d is called the metric on X . If all axioms but the second part of (ii) are satisfied, we call (X, d) a pseudometric space and d is then a pseudometric. Functions $d : X \times X \rightarrow R$ are called distance functions. If a metric d is fixed for a particular discussion, we may drop the ordered-pair notation and simply speak of the "the metric space X " [9].

Definition 2.2: Let X denotes a nonempty set and R , the set of all nonnegative real numbers. Then X together with a function $D : X \times X \times X \rightarrow R$ is called a **D - metric spaces** if it satisfies the following properties:

- i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$.
- ii) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ (symmetry)
- iii) $D(x, y, z) \leq D(x, y, t) + D(x, t, z) + D(t, y, z)$ for

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$x, y, z, t \in X$ (restangle inequality)

A nonempty set X with a D -metric, D is called a D -metric or a generalized metric space and is denoted by (X, D) . Two-metric spaces are defined Gähler as a function. $d : X \times X \times X \rightarrow R$ with properties (ii) and (iii) above and (i) replaced by (i). For distinct points $x, y \in X$ there exists a point $c \in X$ such that $d(x, y, z) \neq 0$ and $d(x, y, z) = 0$ if at least two of x, y and z are equal [4].

Now, using to metric space, lets re-examine Cone Metric Space and Cone Two Metric Space. Lets compare the definitions of Cone Metric Space and Cone Two Metric Space.

Recently, Huang and Zhang introduced cone metric space by generalized the concept of a metric space, replacing the set of real numbers, by on ordered Banach space and obtained some fixed point theorems for contractive mappings. By using both of the concepts, we define a new space cone two-metric space replacing real number in two metric space by on ordered Banach space.

Definition 2.3: Let X be a nonempty set. Assume that the mapping $d : X \times X \rightarrow E$ satisfies:

- i) $d(x, y) \geq 0$ for all $x, y \in X$ on $d(x, y) = 0$ iff $x = y$.
- ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)
- iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

Then d is called a **cone metric** on X and (X, d) is called a **cone metric space**[3].

Definition 2.4: Let X be nonempty set. Suppose the mapping $d : X \times X \times X \rightarrow P$ satisfies :

- i) $d(x, y, z) \geq 0$ for all $x, y, z \in X$ and $d(x, y, z) = 0$ if and only if at least two of x, y, z are equal.
- ii) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ (symmetry)
- iii) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Then d is called a **cone two metric** on X and (X, d) will be called a **cone two metric space**. Cone two metric space will be called normal, if the cone P is normal cone.

Definition 2.5: Let E be a real Banach space and $P \subseteq E$ is be cone. The cone is called normal if there exists a number $K > 0$ such that

$$\forall x, y \in P; 0 \leq x \leq y;$$

implies

$$\|x\| \leq K \|y\|$$

The least number satisfying above is called the normal constant of P .

The normality definition which is above, are valid for "normal cone metric spaces" and "normal cone two metric spaces".

Let's examine a few examples with considering "cone metric space" and "cone two metric space".

Example 1: Let $E = R^2$,
 $P = \{(x, y) \in R^2 \mid x, y > 0\} \subset R^2$,
 $X = R$ and $d : X \times X \rightarrow E$ such that
 $d(x, y) = (|x - y|, \alpha |x - y|)$
 where $\alpha \geq 0$ is a constant. Then (X, d) is cone metric space.

Solution 1: P is cone because;

i) P is closed and $\theta \in P$ to $P \neq \emptyset$. $(x, y) = (0, 1) \in P$ such that $P \neq \{\theta\}$.

ii) Let be $a, b \in R$ where $a, b \geq 0$ and $z = (x_1, y_1)$ and $t = (x_2, y_2) \in P$. Thus

$$x_1, y_1 \geq 0 \text{ and}$$

$$x_2, y_2 \geq 0 \Rightarrow ax_1 \geq 0, ay_1 \geq 0, bx_2 \geq 0, by_2 \geq 0.$$

$$\Rightarrow ax_1 + bx_2 \geq 0 \text{ and } ay_1 + by_2 \geq 0$$

$$\Rightarrow az + bt = a(x_1, y_1) + b(x_2, y_2)$$

$$\Rightarrow (ax_1 + bx_2, ay_1 + by_2) \in P.$$

iii) Let $z = (x, y)$, $-z = (-x, -y) \in P$. Thus

$$x, y \geq 0, -x, -y \leq 0 \Rightarrow x \geq 0, -x \leq 0 \text{ and } y \geq 0, -y \leq 0$$

$$\Rightarrow x = 0 \text{ and } y = 0$$

$$\Rightarrow z = (0, 0) = \theta \in P.$$

Namely, P is cone on E .

Now, show that X - cone metric space:

a) $d(x, y) = (|x - y|, \alpha |x - y|) \in P$ such that $d(x, y) \geq \theta$ and $d(x, y) = \theta$ if and only if $(|x - y|, \alpha |x - y|) = (0, 0) \Leftrightarrow |x - y| = 0$ and $\alpha |x - y| = 0 \Leftrightarrow x = y$ or $\alpha = 0 \Leftrightarrow x = y$.

b) Every $x, y \in R$ to $d(x, y) = (|x - y|, \alpha |x - y|)$
 $= (|y - x|, \alpha |y - x|)$
 $= d(y, x)$

c) $x, y, z \in R$ to $d(x, y) = (|x - y|, \alpha |x - y|)$

$$(|x - z + z - y|, \alpha |x - z + z - y|)$$

$$\leq (|x - z| + |z - y|, \alpha |x - z| + |z - y|)$$

$$\text{So that } d(x, y) \leq d(x, z) + d(z, y).$$

As a result (X, d) is cone metric space.

Example 2: Let $E = R^2$,

$$P = \{(x, y) \in E \mid x, y \geq 0\} \subset R^2.$$

$X = R$ and $d : X \times X \times X \rightarrow E$ and such that

$$d(x, y, z) = (p^n, \alpha p)$$

where

$p = \min(|x - y|, |y - z|, |z - x|)$ and α and n are some fixed positive integers. Then (X, d) is a cone two metric space.

Example 3: Let

$$E = (C_R[0, \infty), \|\cdot\|_\infty)$$

$$P = \{f \in E \mid f(t) \geq 0\}$$

(X, ρ) a metric space and $d : X \times X \rightarrow E$ defined by $d(x, y) = f_{x,y}$ where $f_{x,y}(t) = |x - y|t$.

Then (X, ρ) is a normal cone metric space and normal constant of P is equal to $K = 1$.

Example 4: Let

$E = (C_R[0, \infty), \|\cdot\|_\infty)$, $P = \{f \in E | f(t) \geq 0\}$

(X, ρ) a metric space and $d : X \times X \times X \rightarrow E$ defined by $d(x, y, z) = \rho(x, y, z)\phi$ where $\phi : [0, 1] \rightarrow R^+$ is continuous. Then (X, d) is a normal cone two metric space and the normal constant of P is equal to $K = 1$.

Example 5: Let $E = C_R[0, 1]$ with the supremum norm and $P = \{f \in E | f(t) \geq 0\}$. Then P is a cone two metric space and P is normal cone with normal constant of $K = 1$. Define

$d : X \times X \times X \rightarrow E$ by $d(x, y, z) = |x - y - z|\phi$ where $\phi : [0, 1] \rightarrow R^+$ such that $\phi(t) = e^t$. It is easy to see that d is a cone two metric on X .

Definition 2.6: Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

i) a Cauchy sequence if for every c in B with $c > \theta$, there is N such that for all $n, m > N$, $d(x_n, x_m) < c$;

ii) a convergent sequence if for any $c > \theta$, there is an N such that for all $n > N$, $d(x_n, x) < c$, for some fixed x in X . We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$) [2].

Definition 2.7: Let (X, d) be a cone two metric space with respect to a cone P in a real Banach space E . Let x_n be a sequence in X and $x \in X$. If for every $c \in E$ with $0 < c$ i.e. $c \in \text{int}P$ there is N such that $d(x_n, x, \alpha) < c$ for all $\alpha \in X$ and for all $n > N$. Then $\{x_n\}$ is said to be convergent to X . We denote it by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.

Lemma 2.1: Let (X, d) be a cone two metric space, P be a normal cone with normal constant K . Let $\{x_n\}$ be a sequence in X . Then limit of $\{x_n\}$ is unique if it exists.

Proof: Assume that sequence $\{x_n\}$ converges to two points x and y . For any $c \in \text{int}P$ there is N such that for all $n > N$, $d(x_n, x, \alpha) < c$ and $d(x_n, y, \alpha) < c$ for all $\alpha \in X$. Now,

$$\begin{aligned} d(x, y, \alpha) &\leq d(x, y, x_n) + d(x, x_n, \alpha) + d(x_n, y, \alpha) \\ &\leq c + c + c \\ &\leq 3c \end{aligned}$$

Hence,

$$\|d(x, y, \alpha)\| \leq 3K\|c\|.$$

Since c is arbitrary $d(x, y, \alpha) = 0$ for all $\alpha \in X$.

Definition 2.8: Let E be a real Banach space. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X . If for every $c \in E$ with $0 < c$, there is N such that for all $n, m > N$, $d(x_n, x_m) < c$ then $\{x_n\}$ is called a Cauchy sequence in X [2].

Definition 2.9: Let (X, d) be a cone two metric space. Let

$\{x_n\}$ be a sequence in X . If for every $c \in \text{int}P$ there is a N such that $d(x_n, x_m, \alpha) < c$ for all $\alpha \in X$ and for all $m, n > N$, then $\{x_n\}$ is said to be a Cauchy sequence in X .

Definition 2.10: Let (X, d) be a cone two metric space if every Cauchy sequence is convergent in X the X is said to be a complete cone two metric space.

Lemma 2.2: Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X . If $\{x_n\}$ converges to x , then $\{x_n\}$ is a Cauchy sequence [2].

Proof: For any $c \in E$ with $0 < c$, there is N such that for all $n, m > N$, $d(x_n, x) < c/2$ and $d(x_m, x) < c/2$. Hence $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \leq c/2 + c/2 \leq c$. There for $\{x_n\}$ is a Cauchy sequence.

Lemma 2.3: Let (X, d) be a cone two metric space, $\{x_n\}$ be a sequence in X . If $\{x_n\}$ converges to x , then $\{x_n\}$ is a Cauchy sequence.

Proof: For any $c \in E$ with $0 < c$, there is N such that for all $n, m, \alpha > N$,

$$d(x_n, x_m, x) \leq c/3, d(x_n, x, x_\alpha) \leq c/3, d(x, x_m, x_\alpha) \leq c/3.$$

Hence,

$$\begin{aligned} d(x_n, x_m, x_\alpha) &\leq d(x_n, x_m, x) + d(x_n, x, x_\alpha) + d(x, x_m, x_\alpha) \\ &\leq c/3 + c/3 + c/3 \\ &\leq c \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence.

3 Main Result

We prove some fixed point results for expansion mappings in a cone metric spaces and cone two metric spaces.

Theorem 3.1: Let (X, d) be a complete cone two-metric space, P be a normal cone with normal constant K . Suppose the mapping $T : X \rightarrow X$ satisfies:

$$d(Tx, Ty, \alpha) \leq Kd(x, y, \alpha) + \lambda d(Tx, x, \alpha) + \mu d(Ty, y, \alpha);$$

for all $x, y, \alpha \in X$ for some fixed $K, \lambda, \mu \in [0, 1)$ with $K + \lambda + \mu < 1$.

Then T has a unique fixed point in X and for every $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Corollary 3.1: Let (X, d) be a complete cone two-metric space, P be a normal cone with normal constant K . Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition

$$d(Tx, Ty, \alpha) \leq k.d(x, y, \alpha); \text{ for all } x, y, \alpha \in X$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X . Moreover for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 3.2: Let (X, C) be a complete cone two-metric space. If there exists $\alpha > 1$ constant and an onto self mapping T on X satisfying

$$C(Tx, Ty, Tz) \geq \alpha C(x, y, z)$$

for all $x, y, z \in X$, then T has a unique fixed point [10].

Proof: Let $Tx = Ty$, then $0 = C(Tx, Ty, Tz) \geq \alpha C(x, y, z)$. Hence $C(x, y, z) = 0$, which implies that $x = y$. So T is injective and invertible. If h is the inverse mapping of T , then

$$C(x, y, z) = C(T(hx), T(hy), T(hz)) \geq \alpha C(hx, hy, hz)$$

Hence for all $x, y, z \in X$

$$C(hx, hy, hz) \leq kC(x, y, z), \text{ where } k = 1/\alpha$$

the inverse mapping h has a unique fixed point $u \in X$ i.e. $hu = u$. But $u = T(h(u)) = T(u)$. Thus u is also a fixed point of T . Suppose v is another fixed point of T i.e. $Tv = v$, then $Tv = v = T(h(v)) = h(T(v))$ which implies that Tv is another fixed point of h . Since h has a unique fixed point $u = Tv = v$, hence T has a unique fixed point.

Example 6: Let $E = R, P = \{x \in R, x \geq 0\}$ be a cone. Let $X = [0, \infty)$ with the metric

$$C(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

and the metric

$$d(x, y) = |x - y|.$$

Define the map $T : X \rightarrow X$ by $Tx = 4x, \forall x \in X$. Then

$$C(Tx, Ty, Tz) = d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz)$$

$$= |Tx - Ty| + |Ty - Tz| + |Tx - Tz|$$

$$= |4x - 4y| + |4y - 4z| + |4x - 4z|$$

$$= 4(|x - y| + |y - z| + |x - z|)$$

$$\geq |x - y| + |y - z| + |x - z|$$

$$= d(x, y) + d(y, z) + d(x, z)$$

$$= C(x, y, z)$$

Hence conditions of Theorem 3.2 are satisfied and the point $x = 0$ is the unique fixed point of the mapping T .

Corollary 3.2: Let (X, d) be a complete cone two-metric space, P be a normal cone with normal constant K . For $c \in \text{int}P$ and $x_0 \in X$, set $B(x_0, c) = \{x \in X | d(x_0, x, \alpha) \leq c\}$. Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition $d(Tx, Ty, \alpha) \leq kd(x, y, \alpha)$; for all $x, y \in B(x_0, c)$ and for all $\alpha \in X$ where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in $B(x_0, c)$.

Proof: In view of Corollary 3.1, we only need to prove that $B(x_0, c)$ is complete and $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$. Let x_n be a Cauchy sequence in $B(x_0, c)$. Then this sequence is also a Cauchy sequence in X . As X is complete, $\exists x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. For any $\alpha \in X$ we have

$$d(x_0, x, \alpha) \leq d(x_0, x, x_n) + d(x_0, x_n, \alpha) + d(x_n, x, \alpha)$$

$$\leq d(x_0, x, x_n) + d(x_0, x, \alpha) + c; \text{ for all } \alpha \in X$$

As $x_n \rightarrow x$, we have $d(x_0, x, x_n) = d(x_n, x, \alpha) = 0$.

Therefore $d(x_0, x, \alpha) \leq c$ and $x \in B(x_0, c)$. Thus $B(x_0, c)$ is complete.

Corollary 3.3: Let (X, d) be a complete cone two-metric space, P be a normal cone with normal constant K . Suppose the mapping $T : X \rightarrow X$ with some positive integer n satisfies the contractive condition, that is $d(T^n x, T^n y, \alpha) \leq kd(x, y, \alpha)$; for all $x, y, \alpha \in X$ where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X .

Proof: In view of Corollary 3.1, T^n has a unique fixed point x . Now $(T^n x) = T(T^n x) = Tx$, so Tx is also a fixed point of T^n . As the fixed point of T^n is unique, uniqueness of fixed point of T is followed by the uniqueness of fixed point of T^n .

Corollary 3.4: Let (X, d) be a complete cone two-metric space, P be a normal cone with normal constant K . Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition

$$d(Tx, Ty, \alpha) \leq k(d(Tx, x, \alpha) + d(Ty, y, \alpha)); \text{ for all } x, y, \alpha \in X$$

where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Definition 3.1: Let X be a linear space. A two metric $d : X \times X \times X \rightarrow R^+$ is called translation invariant if $d(x+t, y+t, z+t) = d(x, y, z)$ for all $x, y, z, t \in X$. (1) Observe that (1) is a statement that ties the two metric d with the group structure of the linear space X . Translation invariance is thus a concept which is partly metric and partly algebraic it connects in a particular way the distance function on a linear space with the operation of vector addition. In fact, this connection is tighter than what may seem at first. If you endow a linear space with a translation invariant metric, then, per force, you make the vector addition operation of your space continuous!

Proposition 3.1: Let X be a linear space which is also a metric space. If the metric d of X is translation invariant, then the map $(x, y, z) \rightarrow x + y + z$ is a continuous function from $X \times X \times X$ into X .

Definition 3.2: Let X be a linear space which is also a two metric space. If the two metric d of X is translation invariant and for all convergent $(\lambda_m) \in R^\infty$ and $(x^m) \in X^\infty$, we have

$$\lim \lambda_m x^m = (\lim \lambda_m)(\lim x^m)$$

then X is called a two metric linear space. If in addition, (X, d) is complete then, we say that X is a **complete two metric linear space**. A two metric linear space X is said to be nontrivial if $X \neq \{0\}$, finite dimensional if $\dim(X) < \infty$ and infinite dimensional if $\dim(X) = \infty$.

Proposition 3.2: For any subsets A and B of a two metric linear space X , the following are true:

- $cIX(A+x) = cIX(A) + x$ for all $x \in X$;
- If A is open, then so is $A+B$;
- If A is compact and B is closed, then $A+B$ is closed;

d) If both A and B are compact, so is $A + B$.

Theorem 3.3: Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone two metric d in X such that the cone two metric space (X, d) is complete. Let $f, g : X \rightarrow X$ be a continuous and non-decreasing mapping that is $\subseteq$. Suppose that the following assertions hold:

- i) There exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + 2\beta + 2\gamma < 1$ such that $d(fx, fy, fz) \leq \alpha d(x, y, z) + \beta [d(x, fx, gx) + d(y, fy, gy) + d(z, fz, gz)] + \gamma [d(x, fy, gy) + d(x, fz, gz) + d(y, fx, gx) + d(y, fz, gz) + d(z, fx, gx) + d(z, fy, gy)]$ for all $x, y, z \in X$ with $y, z \subseteq x$.
- ii) There exists $x_0 \in X$ such that $x_0 \subseteq fx_0$
- iii) If an increasing sequence converges to $x \in X$, then $x_n \subseteq x$ for all n .

Corollary 3.5: Cone metric space and Cone Two Metric Space are linearly.

4 Conclusion

In this presentation, "Cone Metric Space and Cone Two Metric Space" has been compared. In 2007, Chinese Mathematicians, Huang and Zhang, defined Cone Metric Space and so they formed the proof and practice of Fixed Point Theorem. Cone Metric Space contains a larger study area through generalization of some terms, definitions and theorems of metric space. In this presentation Cone Two Metric Space which is a more general form of Cone Metric Space has been studied through scrutinizing the general form of metric space, which is Cone Metric Space. Cone Metric Space and Cone Two Metric Space has been compared. Some theorems and proofs about Fixed Point Theorem in the Cone Two Metric Space has been analysed. Cone Metric Space and Cone Two Metric Space on linear term and linear cone two metric space have been researched. It has been seen that Cone Two Metric Space enable the definitions and theorems of metric space to work in a larger area.

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