

An Elementary Beukers-Like Proof for the Irrationality of π^2

F. M. S. Lima^{1,*} and P. G. F. Jordão²

¹ Institute of Physics, University of Brasília, P.O. Box 04455, 70919-970, Brasília-DF, Brazil

² PETMAT, Mathematics Department, University of Brasília, 70910-900, Brasília-DF, Brazil

Received: 06 Jul. 2018, Revised: 10 Dec. 2018, Accepted: 29 Apr. 2019

Published online: 1 Jul. 2019

Abstract: In a pioneering work [Bull. London Math. Soc. **11**, 268 (1979)], Beukers introduced a simple method, based upon double integrals over the unit-square $[0, 1] \times [0, 1]$ involving Legendre polynomials, for proving the irrationality of $\zeta(2)$ and $\zeta(3)$. In a more recent paper [Amer. Math. Monthly **108**, 222 (2001)], Huylebrouck has adapted the Beukers method for proving the irrationality of π , among other mathematical constants. In this note, we modify Huylebrouck's proof in order to show the stronger result that π^2 is irrational using only single integrals over $[0, 1]$.

Keywords: Irrationality proof, Beukers method, Legendre polynomials

1 Introduction

The number π is defined as the ratio of the circumference C of any circle to its diameter $2R$, i.e. $\pi := C/(2R)$. The first proof that π is irrational was presented by Lambert in 1761 [10], when he succeeded in using continued fractions to show that $\tan \theta$ is irrational for all rational $\theta \neq 0$. Since $\tan(\pi/4) = 1$, it follows that $\pi/4$ is irrational. Showing that π^2 is an irrational number is a harder task, so much so that it was completed only in 1808 by Legendre, also by means of continued fractions [11, Note IV]. Much later, Hermite found a nontrivial proof that π^2 is irrational which does not make use of continued fractions [6], a fact revived by Zhou in a recent paper [16]. Interestingly, Zhou also traces a similarity with the celebrated Niven's proof for the irrationality of π [13], which was adapted to π^2 by Iwamoto in 1960 [9]. In Refs. [3, 4, 8, 14] we find some variants involving Hermite-like polynomials, but the 'appearance from nowhere' of these polynomials, together with the difficulty in proving that their coefficients are integers (not all null), make these proofs somewhat artificial. This motivated us to return to a pioneering work by Beukers [1], where a simple method based upon unit-square integrals involving Legendre polynomials is employed to prove the irrationality of $\zeta(2)$ and $\zeta(3)$, $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$, $s > 1$, being the Riemann

zeta function. The successful adaptation of the this method for proving the irrationality of π , as developed by Huylebrouck in a recent paper [7], then suggested that it could be modified into a more direct elementary proof of irrationality for π^2 .

2 A family of integrals of the sine function

We begin redefining π as the smallest real number $b > 0$ that satisfies

$$\int_0^b \sin \theta \, d\theta = 2. \quad (1)$$

This is equivalent to the original definition, as follows from the fact that the above equation reduces to $1 - \cos b = 2$, or $\cos b = -1$, whose smallest positive solution (in radians) is $b = (C/2)/R = C/(2R)$. Now, we choose $f(x) = (1/\pi) \sin(\pi x)$ in Huylebrouck's method [7] and search for a pattern in the integrals $I_k := \int_0^1 x^k f(x) \, dx$. In what follows, $\mathcal{P}_n(x)$ denotes a polynomial of degree n with non-null integer coefficients.

Lemma 1 *Let $k \geq 0$ be an integer. Then $I_k = \frac{1}{\pi} \int_0^1 x^k \sin(\pi x) \, dx = \mathcal{P}_{k/2}(\pi^2)/\pi^{k+2}$ for even values of k and $\mathcal{P}_{(k-1)/2}(\pi^2)/\pi^{k+1}$ otherwise.*

* Corresponding author e-mail: fabio@fis.unb.br

Proof. For $k = 0$, on substituting $\theta = \pi x$ in Eq. (1) one finds

$$I_0 = \frac{1}{\pi} \int_0^1 \sin(\pi x) dx = \frac{1}{\pi} \cdot \frac{2}{\pi} = \frac{2}{\pi^2}. \quad (2)$$

For $k = 1$, an integration by parts promptly yields

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_0^1 x \sin(\pi x) dx \\ &= -\frac{1}{\pi} \left[x \frac{\cos(\pi x)}{\pi} \right]_0^1 + \frac{1}{\pi^2} \int_0^1 \cos(\pi x) dx = \frac{1}{\pi^2}. \end{aligned} \quad (3)$$

For any $k > 1$, two integrations by parts lead to

$$\begin{aligned} I_k &= \frac{1}{\pi} \int_0^1 x^k \sin(\pi x) dx \\ &= \frac{1}{\pi} \left[\frac{1}{\pi} - \frac{k(k-1)}{\pi} I_{k-2} \right] = \frac{1}{\pi^2} - \frac{k(k-1)}{\pi^2} I_{k-2}. \end{aligned} \quad (4)$$

On changing k by $k-2$ in this equation, one has

$$I_{k-2} = \frac{1}{\pi^2} - \frac{(k-2)(k-3)}{\pi^2} I_{k-4}. \quad (5)$$

On substituting this I_{k-2} back in Eq. (4), we find the following recurrence

$$I_k = \frac{1}{\pi^2} - \frac{k(k-1)}{\pi^4} + \frac{k!}{(k-4)!} \frac{1}{\pi^4} I_{k-4}. \quad (6)$$

The repetition of this procedure furnishes

$$\begin{aligned} I_k &= \frac{c_k}{\pi^2} + \frac{c_{k-2}}{\pi^4} + \dots + \frac{c_2}{\pi^k} + \frac{c_0}{\pi^{k+2}} \\ &= \frac{c_0 + c_2 \pi^2 + \dots + c_{k-2} \pi^{k-2} + c_k \pi^k}{\pi^{k+2}}, \end{aligned} \quad (7)$$

for even values of k , and

$$\begin{aligned} I_k &= \frac{c_k}{\pi^2} + \frac{c_{k-2}}{\pi^4} + \dots + \frac{c_3}{\pi^{k-1}} + \frac{c_1}{\pi^{k+1}} \\ &= \frac{c_1 + c_3 \pi^2 + \dots + c_{k-2} \pi^{k-3} + c_k \pi^{k-1}}{\pi^{k+1}}, \end{aligned} \quad (8)$$

for odd values of k . The proof completes by noting that $|c_0| = 2k!$ and $|c_m| = k!/m!$, $0 < m \leq k$, are non-null integers.

□

For instance, $I_2 = (-4 + \pi^2)/\pi^4$, $I_3 = (-6 + \pi^2)/\pi^4$,
 $I_4 = (48 - 12\pi^2 + \pi^4)/\pi^6$, and
 $I_5 = (120 - 20\pi^2 + \pi^4)/\pi^6$.

3 Main Result

With the lemma proved in the previous section in hands, let us prove our main result.

Theorem 1 $\pi^2 \notin \mathbb{Q}$.

Proof. For every integer $n \geq 0$, define $J_n := \frac{1}{\pi} \int_0^1 \widetilde{\mathcal{P}}_n(x) \sin(\pi x) dx$, where

$$\widetilde{\mathcal{P}}_n(x) := \frac{1}{n!} \frac{d^n}{dx^n} [x^n (1-x)^n]$$

is the Legendre polynomial of degree n normalized on the interval $[0, 1]$, whose coefficients are known to be non-null integers [2].¹ As shown in Ref. [7], the advantage of using these polynomials is that they allow for the transference of the n -th derivative to $f(x)$:

$$\begin{aligned} |J_n| &= \frac{1}{\pi} \left| \int_0^1 \frac{1}{n!} \frac{d^n}{dx^n} [x^n (1-x)^n] \sin(\pi x) dx \right| \\ &= \frac{1}{\pi n!} \left| (-1)^n \int_0^1 x^n (1-x)^n \frac{d^n}{dx^n} \sin(\pi x) dx \right| \\ &= \frac{1}{\pi n!} \left| \int_0^1 x^n (1-x)^n \pi^n \begin{Bmatrix} \pm \sin(\pi x) \\ \pm \cos(\pi x) \end{Bmatrix} dx \right|. \end{aligned} \quad (9)$$

Since the function $x^n (1-x)^n = [x(1-x)]^n$ is symmetrical with respect to $x = 1/2$, then $J_n = 0$ for all odd values of n , due to the factor $\cos(\pi x)$. For even values of n , on the other hand, both $x(1-x) > 0$ and $\sin(\pi x) > 0$ for all $x \in (0, 1)$, which implies that

$$|J_n| = \frac{\pi^{n-1}}{n!} \int_0^1 [x(1-x)]^n \sin(\pi x) dx > 0. \quad (10)$$

On taking $M := \max_{x \in [0, 1]} \{x(1-x)\}$ into account for determining an upper bound for $|J_n|$, one finds

$$0 < |J_n| \leq \frac{\pi^{n-1}}{n!} M^n \int_0^1 \sin(\pi x) dx. \quad (11)$$

Since $M = 1/4$, which is attained at $x = 1/2$, and $\int_0^1 \sin(\pi x) dx = 2/\pi$, as seen in Eq. (2), then

$$0 < |J_n| \leq \frac{\pi^{n-1}}{n!} \left(\frac{1}{4}\right)^n \frac{2}{\pi} = \frac{2}{\pi^2 n!} \frac{\pi^n}{4^n}. \quad (12)$$

Now, assume, towards a contradiction, that $\pi^2 \in \mathbb{Q}$. From Lemma 1, this would imply that $J_n \in \mathbb{Q} \setminus \{0\}$ for every n even, since the integrals J_n are linear combinations (with integer coefficients) of the integrals I_k , $k \leq n$. Then, $|J_n| = A_n/B_n$, with A_n and B_n being both positive integers. From Eq. (12), one has

$$0 < \frac{A_n}{B_n} \leq \frac{2}{\pi^2 n!} \left(\frac{\pi}{4}\right)^n. \quad (13)$$

Since n is even, we can make $n = 2\ell$, $\ell \geq 0$ being an integer, and write $\pi^2 = a/b$, a and b being positive integers, which changes the above inequality to

$$0 < A_{2\ell} \leq 2 \frac{b}{a} \frac{B_{2\ell}}{(2\ell)!} \left(\frac{\pi}{4}\right)^{2\ell}. \quad (14)$$

¹ For instance, $\widetilde{\mathcal{P}}_0(x) = 1$, $\widetilde{\mathcal{P}}_1(x) = 1 - 2x$, $\widetilde{\mathcal{P}}_2(x) = 1 - 6x + 6x^2$, and $\widetilde{\mathcal{P}}_3(x) = 1 - 12x + 30x^2 - 20x^3$.

According to Lemma 1, the denominator of $I_{2\ell}$ is $\pi^{2\ell+2} = (\pi^2)^{\ell+1} = (a/b)^{\ell+1}$, so $B_{2\ell} = a^{\ell+1}$. Then

$$0 < A_{2\ell} \leq 2 \frac{b}{a} \frac{a^{\ell+1}}{(2\ell)!} \left(\frac{\pi^2}{16} \right)^\ell. \quad (15)$$

At last, since $\pi^2/16 < 10/16 = 5/8 < 1$, the above inequality reduces to

$$0 < A_{2\ell} < 2b \frac{a^\ell}{(2\ell)!} \left(\frac{5}{8} \right)^\ell. \quad (16)$$

Since both $a^\ell/(2\ell)!$ and $(5/8)^\ell$ tend to 0 as $\ell \rightarrow \infty$, then for large enough ℓ we would have $0 < A_{2\ell} < 1$, which is impossible because $A_{2\ell}$ was assumed to be an integer. $\square$

4 Conclusion

We have presented in this note a Beukers-like proof of irrationality for π^2 . The transcendence of π , proved by Lindemann in 1882 [12], of course guarantees that π^n is irrational for all integers $n > 0$, but the interest in those irrationality proofs is more related to the investigation of irrationality of other real numbers such as the Euler-Mascheroni constant $\gamma := \lim_{n \rightarrow \infty} H_n - \ln n$, $\ln \pi$, Catalan's constant $G := \sum_{n \geq 0} (-1)^n / (2n+1)^2$, and $\zeta(5)$, for which unit-square integral representations are already known [5, 15]. We hope the proof put forward here can be useful for those researchers who are investigating the irrationality of such numbers.

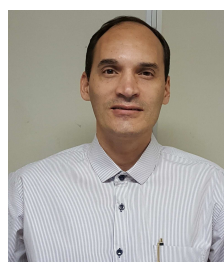
Acknowledgement

PGFJ thanks PETMAT-UnB and FNDE for financial support in the form of a scholarship.

References

- [1] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$* , Bull. London Math. Soc. **11** (1979) 268–272.
- [2] F. Beukers, *Legendre polynomials in irrationality proofs*, Bull. Austral. Math. Soc. **22** (1980) 431–438.
- [3] D. Desbrow, *On the Irrationality of π^2* , Amer. Math. Monthly **97** (1990) 903–906.
- [4] T. Estermann, *A theorem implying the irrationality of π^2* , J. London Math. Soc. **41** (1966) 415–416.
- [5] J. Guillera and J. Sondow, *Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent*, Ramanujan J., **16** (2008), 247–270.
- [6] C. Hermite, *Sur quelques approximations algébriques*, J. de Crelle **76** (1873) 342–344.

- [7] D. Huylebrouck, *Similarities in irrationality proofs for π , $\ln 2$, $\zeta(2)$, and $\zeta(3)$* , Amer. Math. Monthly **108** (2001) 222–231.
- [8] K. Inkeri, *The irrationality of π^2* , Nordisk Mat. Tidskr. **8** (1960) 11–16.
- [9] Y. Iwamoto, *A proof that π^2 is irrational*, J. Osaka Inst. Sci. Tech. **1** (1949) 147–148.
- [10] J. Lambert, *Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques*, Histoire Acad. Roy. des Sciences et Belles Lettres, Berlin, Année 1761 (1768), 265–276.
- [11] A. M. Legendre, *Éléments de géométrie, avec des notes*, 7me ed., Didot, Paris, 1808.
- [12] F. Lindemann, *Über die zahl π* , Math. Ann. **20** (1882), 213–225.
- [13] I. Niven, *A simple proof that π is irrational*, Bull. Amer. Math. Soc. **53** (1947) 509.
- [14] E. M. Schröder, *Zur Irrationalität von π^2 und π* , Mitt. Math. Ges. Hamburg **13** (1993) 249.
- [15] E. W. Weisstein, *Unit-square Integrals*. From MathWorld, a Wolfram Web Resource. Available at: <http://mathworld.wolfram.com/UnitSquareIntegral.html>
- [16] L. Zhou, *Irrationality proofs á la Hermite*, Math. Gaz. **95** (2011) 407–413.



Fábio M. S. Lima is an Assistant Professor of Physics at University of Brasília. He received the PhD degree in Physics at University of Brasília in 2003. His research interests in mathematics are in Analytic Number Theory, particularly the properties of the transcendental functions

related to the Gamma and Riemann zeta functions, as well as irrationality and transcendence proofs for numbers related to e , π , odd zeta values, and Catalan's constant G . He has published research articles in reputed international journals of both theoretical physics and mathematical sciences. He is referee of a number of physics and mathematics journals.



Pedro G. F. Jordão is a graduate student of mathematics at University of Brasília. He is a fellow of the 'Programa de Pós-Graduação' (PPG-UnB), supported by CNPq. In the past two years, as an undergraduate student, he was a fellow of the tutoring

project PETMAT-UnB, supported by FNDE, developing a research program in number theory under the supervision of professor F. M. S. Lima. His main research interests are in Number Theory and Mathematical Analysis.