Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/070201

On Normality Conditions of p-Hyponormal Operators

Mohammed Shehu Shagari

Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Nigeria

Received: 3 Mar. 2019, Revised: 28 Apr. 2019, Accepted: 1 May 2019
Published online: 1 Jul. 2019

Abstract: In this paper, certain classes of non-normal operators, namely p-hyponormal and log-hyponormal operators are studied. In particular, some conditions implying normality of these operators are investigated.

Keywords: Aluthge transform ; polar decomposition; p-hyponormal operators ; log-hyponormal operators; partial isometry.

1 Introduction

Let $B(H)$ be the algebra of all bounded linear operators on H, the Hilbert space. It is known that every operator T on H can be factorized into $T = U|T|$, where U is a partial isometry. Denote the null space of an operator U by $N(U)$. If U is determined in one and only one way under the condition $N(U) = N(|T|)$, then this factorization is called the *polar decomposition of* T. Putting $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*T] = T^*T - TT^* = |T|^2 - |T^*|^2$, known as the self commutator of T, the operator T is said to be normal if its self commutator equals zero, p-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. An invertible p-hyponormal operator T is said to be log-hyponormal if it satisfies $\log(T^*T) \geq \log(TT^*)$ (see [5]). If $p = 1$, then T is called hyponormal and semi-hyponormal for $p = \frac{1}{2}$. Obviously, T is hyponormal if and only if $\|Tx\| = \|T^*x\|$, for any x in H. In general,we have the inclusion $normal \subseteq hyponormal \subseteq p-hyponormal$ operators.

Throughout this paper , the set of all p-hyponormal operators is denoted by $P-H$. The expression $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where U is unitary, is known as the Aluthge transform (see [1]). For some results on generalized Aluthge transform, see [9] and the reference therein.

In operator theory, one commonly investigated problem is the normality of operators. A particular article in this direction is that of Moslehian and Nabavi [6]. They showed, among other things that for $T \in P-H$,with the polar decomposition $T = U|T|$ such that $U^{n_0} = U^*$ for a nonnegative integer n_0, the operator T is normal. Analogously, Rashid [8] proved that for a cartesian decomposition $T = X + iY \in B(H)$ of the operator T, where XY is p-hyponormal, if X or Y is positive, then, T is normal. For similar results and generalizations, see [1,2,4,5,6,11].

In this paper, using the modified Aluthge transform studied by Shagari et al [7], given as $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$, $k,m \in \mathbb{N}$ with $k \leq m$, where U is a partial isometry, we study an order property of hyponormal operators and subsequently present some normality conditions of p-hyponormal operators. Our results extend the works presented in [1,12] and [6, Theorem 2.1] as well as a few related articles existing in the literature.

2 Preliminaries

In the sequel, we require the applications of the following results:

Theorem 1.*[12](Löwner-Heinz Inequality) Let $A,B \in B(H)$, where H is a complex Hilbert space. If $A \geq B \geq 0$ and $\alpha \in [0,1]$,then*

$$A^\alpha \geq B^\alpha \geq 0. \tag{1}$$

Lemma 1.*[6] Let $T \in B(H)$, where H is a complex Hilbert space and $T = U|T|$ be the polar decomposition of T. Then, the following hold:*

(i)$N(|T|) = N(T)$,

* Corresponding author e-mail: shagaris@ymail.com

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(ii)$|T^*|^n = U|T|^nU^*$,
(iii)$|T|^n = U^*|T|^nU$, for any positive integer n.

*Remark.*This result is also true for any positive real.

Lemma 2.*[11](Furuta Inequality)*
Let H be a complex Hilbert space and $A,B \in B(H)$. If $A \geq B \geq 0$, then the following inequalities:

(i)$(B^rA^pB^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$
(ii)$A^{\frac{p+2r}{q}} \geq (A^rB^pA^r)^{\frac{1}{q}}$
hold for $p \geq 0$, $q \geq 1$, $r \geq 0$ with $(1+2r)q \geq p+2r$.

Lemma 3.*[10](Hansen Inequality) Let H be a complex Hilbert space and $A,B \in B(H)$. If $A \geq 0$, $0 \leq \alpha \leq 1$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\alpha \geq B^*A^\alpha B. \tag{2}$$

3 Main Results

In the beginning of this section, as a prerequisite to establishing the fist result on normality conditions of the p-hyponormal operators T, an order property of T is presented and proved for completeness. This result refines the work of [12] on powers of p-hyponormal operators.

Theorem 2.*Let $T \in P-H$ with the polar decomposition $T = U|T|$, where the operator U is a partial isometry. Then, for $n \geq 0$, $n \geq p$, the following inequalities hold:*

$$\begin{aligned}\left(T^{n+1^*}T^{n+1}\right)^{\frac{p}{n+1}} &\geq \cdots \geq \left(T^*T\right)^p \\ &\geq \left(TT^*\right)^p \geq \cdots \\ &\geq \left(T^{n+1}T^{n+1^*}\right)^{\frac{p}{n+1}}.\end{aligned} \tag{3}$$

*Proof.*Let $T = U|T|$ and $T^* = U^*|T^*|$ be the polar decomposition of T and T^*, respectively.
Put

$$A_{n+1} = \left(T^{n+1^*}T^{n+1}\right)^{\frac{p}{n+1}}, B_{n+1} = \left(T^{n+1}T^{n+1^*}\right)^{\frac{p}{n+1}}, n \geq 0. \tag{4}$$

Then,

$$A_{n+1} = |T^{n+1}|^{\frac{2p}{n+1}} = U^*|T^{n+1}|^{\frac{2p}{n+1}}U$$

$$B_{n+1} = |T^{n+1^*}|^{\frac{2p}{n+1}} = U|T^{n+1}|^{\frac{2p}{n+1}}U^*.$$

Inductively, it suffices to show that

$$A_{n+1} \geq A_1 \geq B_1 \geq B_{n+1}. \tag{5}$$

For $n = 0$, the inequalities clearly hold. Setting

$$A_1 = |T|^{2p} \geq |T^*|^{2p} = B_1, \tag{6}$$

we have

$$A_1 = U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^* = B_1. \tag{7}$$

From above equations, using the induction hypothesis, we get

$$U^*A_kU \geq U^*A_1U \geq A_1 \tag{8}$$

Now, using Lemma 3, let

$$C_{k+1} = \left(U^*A_{k+1}^{\frac{k+1}{p}}U\right)^{\frac{p}{k+1}} \geq U^*A_{k+1}U \geq U^*A_kU \geq A_1. \tag{9}$$

To show that the result is true for $n = k+1$, consider

$$\begin{aligned}A_{k+2} &= \left(T^{k+2*}T^{k+2}\right)^{\frac{p}{k+2}} \\ &= \left(T^*\left(T^{k+1*}T^{k+1}\right)T\right)^{\frac{p}{k+2}} \\ &= \left(|T|U^*A_{k+1}^{\frac{k+1}{p}}U|T|\right)^{\frac{p}{k+2}} \\ &= \left(A_1^{\frac{1}{2p}}U^*A_{k+1}^{\frac{k+1}{p}}UA_1^{\frac{1}{2p}}\right)^{\frac{p}{k+2}} \\ &= \left(A_1^{\frac{1}{2p}}C_{k+1}^{\frac{k+1}{p}}A_1^{\frac{1}{2p}}\right)^{\frac{p}{k+2}} \\ &\geq A_1^{(\frac{1}{2p}+\frac{k+1}{p}+\frac{1}{2p})(\frac{p}{k+2})} = A_1,\end{aligned}$$

by Lemma 2, where $r' = \frac{1}{2p}$, $p' = \frac{k+1}{p}$, $q' = \frac{k+2}{p}$.
Similarly,

$$\begin{aligned}B_{k+2} &= \left(T^{k+2}T^{k+2*}\right)^{\frac{p}{k+2}} \\ &= \left(U|T|B_{k+1}^{\frac{k+1}{p}}|T|U^*\right)^{\frac{p}{k+2}} \\ &= U\left(A_1^{\frac{1}{2p}}B_{k+1}^{\frac{k+1}{p}}A_1^{\frac{1}{2p}}\right)U^* \\ &\leq UA^{(\frac{1}{2p}+\frac{k+1}{p}+\frac{1}{2p})(\frac{p}{k+2})} \\ &= UA_1U^* = B_1.\end{aligned}$$

Consequently,

$$A_{k+2} \geq A_2 \geq A_1 \geq B_1 \geq B_2 \geq B_{k+2}. \tag{10}$$

Therefore, by the induction hypothesis, the result holds for all $n \geq 0$. This completes the proof that the operator T^{n+1} is $\left(\frac{p}{n+1}\right)$-hyponormal.

Corollary 1.*Let $T \in P-H$ with the polar decomposition $T = U|T|$, where U is a partial isometry. Then, for $n,m \geq 0$, the following inequalities hold:*

$$\begin{aligned}\left(T^{n+1^*}T^{n+1}\right)^{\frac{p}{(n+1)^{m+1}}} &\geq \cdots \geq (T^*T)^{\frac{p}{(n+1)^m}} \\ &\geq (TT^*)^{\frac{p}{(n+1)^m}} \geq \cdots \\ &\geq \left(T^{n+1}T^{n+1^*}\right)^{\frac{p}{(n+1)^{m+1}}}\end{aligned}$$

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*Proof.*Let T be a p-hyponormal operator with the polar decomposition $T = U|T|$. Then, for $n \geq 0$, Theorem 2 holds. That is, for $n \geq p$,

$$\left(T^{n+1^*}T^{n+1}\right)^{\frac{p}{n+1}} \geq \cdots \geq \left(T^*T\right)^p \geq \left(TT^*\right)^p \geq \cdots \geq \left(T^{n+1}T^{n+1^*}\right)^{\frac{p}{n+1}}. \tag{11}$$

Applying Löwner-Heinz inequality m-times to the above expression with $\alpha = \frac{1}{n+1} \in (0,1]$, we obtain the result.

3.1 Conditions Implying normality

Our first result here is a consequence of Theorem 2.

Corollary 2.*Let the operator T be p-hyponormal. If T^{n+1} is normal, then, T is normal.*

Proof. Let $T \in P-H$. Then, from Theorem 2 the operator T^{n+1} is $(\frac{p}{n+1})$-hyponormal (i).
Since T^{n+1} is normal, then,

$$T^{n+1^*}T^{n+1} = T^{n+1}T^{n+1^*} \qquad (ii)$$

From (i) and (ii), we have

$$\left(T^{n+1^*}T^{n+1}\right)^{\frac{p}{n+1}} = (T^*T)^p$$
$$= (TT^*)^p = \left(T^{n+1}T^{n+1^*}\right)^{\frac{p}{n+1}}$$

Therefore, $T^*T = TT^*$; proving the normality of T.

Theorem 3.*Let $T \in P-H$ with the polar decomposition $T = U|T|$ such that $U^n = U^{m*}$, $U^{n+n*} \longrightarrow I$ as $n \longrightarrow \infty$ and $N(T) = N(T^*)$, for all natural numbers $n,m \geq 1$. Then, the operator T is normal.*

Proof. First, recall that the hypothesis $N(T) = N(T^*)$ is equivalent to $N(T)^\perp = N(T^*)^\perp$. Since $N(T)^\perp$ and $N(T^*)^\perp$ are initial and final subspaces of the Hilbert space H, then $N(T)^\perp = N(T^*)^\perp$ implies that the initial and final projections coincide. That is,

$$U^*U = UU^* \tag{12}$$

Now, let $T \in B(H)$ be a p-hyponormal operator for $p > 0$. This means

$$(T^*T)^p \geq (TT^*)^p$$

From which we have

$$|T|^{2p} \geq |T^*|^{2p} = U|T|^{2p}U^*. \tag{13}$$

By pre-multiplying and post-multiplying the above equation by U and U^* respectively, we have

$$U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*}. \tag{14}$$

Applying the same technique gives

$$U^2|T|^{2p}U^{2*} \geq U^3|T|^{2p}U^{3*}. \tag{15}$$

Consequently, we obtain

$$|T|^{2p} \geq U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*} \geq U^3|T|^{2p}U^{3*}. \tag{16}$$

Continuing in this fashion, generates the following operator inequalities for $n,m \geq 1$:

$$|T|^{2p} \geq U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*} \geq U^3|T|^{2p}U^{3*} \geq \cdots \geq U^{n+m}|T|U^{(n+m)*}. \tag{17}$$

We observe that

$$U^{n+m} = U^nU^m = U^{m*}U^m = U^{(n+m)*}. \tag{18}$$

For $m = 1$, we have $U^{m*}U^m = U^*U$, which is the initial projection onto the initial subspace of H.
Now, we will use the Principle of Mathematical induction to show that $U^{m*}U^m$ is a projection operator. For this, suppose equation (18) holds for $m = k$. Then, for $m = k+1$, it is enough to prove that $U^{(k+1)*}U^{k+1}$ is self-adjoint and idempotent. Thus, we consider:

$$\begin{aligned}\left(U^{(k+1)*}U^{k+1}\right)^* &= U^{(k+1)*}\left(U^{(k+1)*}\right)^* \\ &= U^{(k+1)*}U^{(k+1)**} \\ &= U^{(k+1)*}U^{(k+1)}.\end{aligned} \tag{19}$$

This shows that $U^{(k+1)*}U^{k+1}$ is self-adjoint. Moreover,

$$\begin{aligned}\left(U^{(k+1)*}U^{k+1}\right)^2 &= U^{(k+1)*}U^{k+1}U^{(k+1)*}U^{k+1} \\ &= \left(U^{k+1}\right)^*U^kU\left(U^{k+1}\right)^*U^kU \\ &= \left(U^{k+1}\right)^*U^kUU^*U^{k*}U^kU \\ &= U^{(k+1)*}U^kU^*UU^{k+k*}U \\ &= U^{(k+1)*}U^kU^{k+k*}U \\ &= U^{(k+1)*}U^kU^{n+n*}U \\ &= U^{(k+1)*}U^kIU \\ &= U^{(k+1)*}U^kU = U^{(k+1)*}U^{k+1}.\end{aligned}$$

This proves that $U^{(k+1)*}U^{k+1}$ is idempotent. From (18) and (19), it follows that
$U^{n+m} = U^{m+m*} = U^{(n+m)*}$ is a projection operator. Therefore

$$U^{(n+m)}|T|^{2p}U^{(n+m)*} = |T|^{2p}. \tag{20}$$

Hence, we obtain

$$|T|^{2p} = U^{(n+m)}|T|^{2p}U^{(n+m)*}$$

This gives $|T|^2 = |T^*|^2$. Since $(T^*T)^{\frac{1}{2}} = |T|$ and $(TT^*)^{\frac{1}{2}} = |T^*|$, then, we have
$T^*T = TT^*$, proving the normality of T.

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*Remark.*In Theorem 3, if the operator T is log-hyponormal, the proof is basically the same following the definition $\log(T^*T) \geq log(TT^*)$ together with the fact that $\log|T| = U^*(\log|T|)U^*$ and $\log|T^*| = U(\log|T|)U^*$.

Corollary 3.*(see [6]) Let $T \in P-H$ with the polar decomposition $T = U|T|$ such that $U^{n_0} = U^*$ for some $n_0 \in \mathbb{N}$. Then T is normal.*

Theorem 4.*Let $T \in P-H$ with the polar decomposition $T = U|T|$, where U is a partial isometry. Then, for any $k,m,n \in \mathbb{N}$, with $k \leq m$, the following statements are equivalent:*

(a)T^{n+1} is normal,
(b)T is normal,
(c)$\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$ is normal .

*Proof.*Let the operator T be p-hyponormal with the polar decomposition $T = U|T|$.

(a) implies (b)
Suppose T^{n+1} is normal. Since the hypotheses of Theorem 2 are satisfied, then, the normality of T^{n+1} implies that T is normal. That is, $T^*T = TT^*$.

(b) implies (c)
Assume T is normal. Then, we observe that

$$\begin{aligned}
\widetilde{T^*}\widetilde{T} &= \left(|T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{2m}}\right)\left(|T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}\right)\\
&= |T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{m}}U|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}|T|^{\frac{k}{m}}|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}\left((T^*T)^{\frac{1}{2}}\right)^{\frac{k}{m}}|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}\left((TT^*)^{\frac{1}{2}}\right)^{\frac{k}{m}}|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}|T^*|^{\frac{k}{m}}|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}U|T|^{\frac{k}{m}}U^*|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}|T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{2m}}\\
&= \left(|T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}\right)\left(|T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{2m}}\right) = \widetilde{T}\widetilde{T^*}.
\end{aligned}$$

That is, $\widetilde{T^*}\widetilde{T} = \widetilde{T}\widetilde{T^*}$. This proves that $\widetilde{T}$ is normal.

Conversely, (c) implies (a)
Let $\widetilde{T}$ be normal. Then, we have to show that T^{n+1} is normal. For this, consider $\widetilde{T^*}\widetilde{T} = \widetilde{T}\widetilde{T^*}$. That is,

$$|T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{m}}U|T|^{\frac{k}{2m}} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{m}}U^*|T|^{\frac{k}{2m}}$$

Or $U^*|T|^{\frac{k}{m}}U = U|T|^{\frac{k}{m}}U^*$. This gives $|T|^{\frac{k}{m}} = |T^*|^{\frac{k}{m}}$. From which we have $|T| = |T^*|$ or $(T^*T)^{\frac{1}{2}} = (TT^*)^{\frac{1}{2}}$. Consequently,

$$T^*T = TT^* \tag{21}$$

Pre-multiplying (21) by T^n to obtain

$$T^{n+1^*}T = T^{n+1}T^*. \tag{22}$$

Again, post-multiplying the above equation by T^n, yields

$$T^{n+1^*}T^{n+1} = T^{n+1}T^{n+1^*}. \tag{23}$$

This shows that T^{n+1} is normal.

Corollary 4.*Every p-hyponormal operator $T \in P-H$ is normal if and only if for any $k,m \in \mathbb{N}$, with $k \leq m$, $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$ is normal.*

Corollary 5.*Every $\left(\frac{p}{n+1}\right)$-hyponormal operator T is T^{n+1} normal if and only if for any $k,m,n \in \mathbb{N}$, with $k \leq m$, $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$ is normal.*

*Proof.*The proofs of Cor.4 and Cor.5 follow immediately from Theorem 2.

Theorem 5.*Let $T \in P-H$ with the polar decomposition $T = U|T|$ and $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}, k,m \in \mathbb{N}$, with $k \leq m$. If $\widetilde{T}$ is normal, then, the operator T is normal.*

*Proof.*Let T be p-hyponormal. Then, $(T^*T)^p \geq (TT^*)^p$ implies $|T|^{2p} \geq |T^*|^{2p}$. From which we have

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^* \tag{24}$$

Since $\widetilde{T}$ is normal, then

$$\begin{aligned}
\widetilde{T^*}\widetilde{T} &= |T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{2m}}\cdot|T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}\\
&= |T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{m}}U|T|^{\frac{k}{2m}} = \widetilde{T}\widetilde{T^*}.
\end{aligned}$$

In what follows, we employ the techniques of [5, Theorem 9] as follows:
Let $b = |T|^{\frac{k}{2m}}a \in R\left(|T|^{\frac{k}{2m}}\right)$, where $R(M)$ is the range set of M. Then,

$$\left\langle |T|^{\frac{k}{2m}}U|T|^{\frac{k}{m}}U^*|T|^{\frac{k}{2m}}a,a\right\rangle = \left\langle |T|^{\frac{k}{2m}}U^*|T|^{\frac{k}{m}}U|T|^{\frac{k}{2m}}a,a\right\rangle$$
$$\left\langle U|T|^{\frac{k}{m}}U^*|T|^{\frac{k}{2m}}a,|T|^{\frac{k}{2m}}a\right\rangle = \left\langle U^*|T|^{\frac{k}{m}}U|T|^{\frac{k}{2m}}a,|T|^{\frac{k}{2m}}a\right\rangle$$
$$\left\langle U|T|^{\frac{k}{m}}U^*b,b\right\rangle = \left\langle U^*|T|^{\frac{k}{m}}Ub,b\right\rangle$$

Let $y \in N(|T|) = N(T) = N(U)$. Since $N(T) \subseteq N(T^*) = N(U^*)$, we get
$U|T|^{\frac{k}{m}}U^*y = 0$ and $U^*|T|^{\frac{k}{m}}U = 0$. It follows that

$$\left\langle U|T|^{\frac{k}{m}}U^*(b+y),(b+y)\right\rangle = \left\langle U^*|T|^{\frac{k}{m}}U(b+y),(b+y)\right\rangle$$

and $U|T|^{\frac{k}{m}}U^* = U^*|T|^{\frac{k}{m}}U$.
Suppose $N(T) = N(T^*)$. Put $T^*a = 0$. Then, $|T|U^*a = 0$. This means
$U^*a \in N(|T|) = N\left(|T|^{\frac{k}{m}}\right)$. Thus,

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$U|T|^{\frac{k}{m}}U^*a = U^*|T|^{\frac{k}{m}}Ua = 0$. Therefore, $|T|^{\frac{k}{m}}Ua = 0$; which means

$$Ua \in N\left(|T|^{\frac{k}{m}}\right) = N(|T|) = N(T) \subseteq N(T^*) = N(U^*)\cdot$$

Hence, $U^*Ua = 0$, implies $Ua = 0$. This shows that $a \in N(U) = N(T)$ and $N(T) = N(T^*)$.
Let $P = U^*U$ be the initial projection of H having range $\overline{R(U)} = \overline{R(U^*)}$. then, $U^*U = UU^*$. From which we have

$$U\left(|T|^{\frac{k}{m}}\right)^2 U^* = U^*\left(|T|^{\frac{k}{m}}\right)^2 U. \tag{25}$$

Using the p-hyponormality of T, we get

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*$$

Hence,

$$U^*|T|^{2p}U = |T|^{2p} = U|T|^{2p}U^*\cdot$$

and

$$|T|^{2p}U = U|T|^{2p}U^*U = U|T|^{2p}\cdot$$

This gives $|T|U = U|T|$. Therefore,

$$\begin{aligned} TT^* = |T|U^*U|T| &= |T|^2 \\ &= |T|^2U^*U = |T|^2UU^* \\ &= U|T|^2U^* = TT^*; \end{aligned}$$

that is, T and T^* commute, proving the normality of T.

Conclusion

This work presents a study on the order properties of p-hyponormal operators. The idea on the order property is extended to investigate some normality characterizations of the operator T. Thereafter, a few novel hypotheses implying normality are discussed.

Aknowledgement

The author is grateful to the editors and the anonymous referees for careful checking of the details and for their helpful comments to improve this paper.

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Mohammed Shehu Shagari received an MSc degree in Mathematics from COMSATS Institute of Information Technology, Pakistan. Currently, he is a PhD research scholar at COMSATS University, Pakistan and a faculty member at Ahmadu Bello University, Nigeria. His research interests are in the areas of Operator theory, Variational inequalities and Fixed point theory.

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