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ON THE DENSITY OF RANGES OF GENERALIZED DIVISOR FUNCTIONS WITH RESTRICTED DOMAINS

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Dedicated to Professor Harald Niederreiter on the occasion of his 70th birthday

ABSTRACT. We begin by defining functions $\sigma_{t,k}$, which are generalized divisor functions with restricted domains. For each positive integer k, we show that, for $r > 1$, the range of $\sigma_{-r,k}$ is a subset of the interval $\left[1, \dfrac{\zeta(r)}{\zeta((k+1)r)}\right)$. After some work, we define constants η_k which satisfy the following: If $k \in \mathbb{N}$ and $r > 1$, then the range of the function $\sigma_{-r,k}$ is dense in $\left[1, \dfrac{\zeta(r)}{\zeta((k+1)r)}\right)$ if and only if $r \leq \eta_k$. We end with an open problem.

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1. Introduction

Throughout this paper, we will let $\mathbb{N}$ denote the set of positive integers, and we will let $\mathbb{P}$ denote the set of prime numbers. We will also let p_i denote the i^{th} prime number.

For a real number t, define the function $\sigma_t \colon \mathbb{N} \to \mathbb{R}$ by $\sigma_t(n) = \displaystyle\sum_{\substack{d|n \\ d>0}} d^t$ for all $n \in \mathbb{N}$. Note that σ_t is multiplicative for any real t. For each positive integer

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n, if $r > 1$, we have $1 \le \sigma_{-r}(n) = \sum_{\substack{d|n \\ d>0}} \frac{1}{d^r} < \sum_{i=1}^{\infty} \frac{1}{i^r} = \zeta(r)$, where ζ denotes the Riemann zeta function. The author has shown [1] that if $r > 1$, then the range of the function σ_{-r} is dense in the interval $[1, \zeta(r))$ if and only if $r \le \eta$, where η is the unique number in the interval $(1, 2]$ that satisfies the equation $\left(\frac{2^\eta}{2^\eta - 1}\right)\left(\frac{3^\eta + 1}{3^\eta - 1}\right) = \zeta(\eta)$.

For each positive integer k, let S_k be the set of positive integers defined by

$$S_k = \{n \in \mathbb{N} : p^{k+1} \nmid n \; \forall \, p \in \mathbb{P}\}.$$

For any real number t and positive integer k, let $\sigma_{t,k} \colon S_k \to \mathbb{R}$ be the restriction of the function σ_t to the set S_k, and let $\log \sigma_{t,k} = \log \circ \sigma_{t,k}$. We observe that, for any $k \in \mathbb{N}$ and $r > 1$, the range of $\sigma_{-r,k}$ is a subset of $\left[1, \frac{\zeta(r)}{\zeta((k+1)r)}\right)$. This is because, if we allow $\prod_{i=1}^{v} q_i^{\beta_i}$ to be the canonical prime factorization of some positive integer in S_k (meaning that $\beta_i \le k$ for all $i \in \{1, 2, \ldots, v\}$), then

$$1 = \sigma_{-r,k}(1) \le \sigma_{-r,k}\left(\prod_{i=1}^{v} q_i^{\beta_i}\right) = \prod_{i=1}^{v} \sigma_{-r,k}(q_i^{\beta_i}) = \prod_{i=1}^{v}\left(\sum_{j=0}^{\beta_i} q_i^{-jr}\right)$$

$$\le \prod_{i=1}^{v}\left(\sum_{j=0}^{k} q_i^{-jr}\right) < \prod_{i=1}^{\infty}\left(\sum_{j=0}^{k} p_i^{-jr}\right) = \prod_{i=1}^{\infty} \frac{1 - p_i^{-(k+1)r}}{1 - p_i^{-r}} = \frac{\zeta(r)}{\zeta((k+1)r)}.$$

To simplify notation, we will write $G_k(r) = \frac{\zeta(r)}{\zeta((k+1)r)}$.

Our goal is to analyze the ranges of the functions $\sigma_{-r,k}$ in order to find constants analogous to η for each positive integer k. More formally, for each $k \in \mathbb{N}$, we will find a constant η_k such that if $r > 1$, then the range of $\sigma_{-r,k}$ is dense in $[1, G_k(r))$ if and only if $r \le \eta_k$.

2. The Ranges of $\sigma_{-r,k}$

Definition 2.1. For $k, m \in \mathbb{N}$ and $r \in (1, \infty)$, let

$$f_k(m, r) = \log\left(1 + \frac{1}{p_m^r}\right) + \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right).$$

Notice that, for any $k \in \mathbb{N}$ and $r \in (1, \infty)$, the range of $\sigma_{-r,k}$ is dense in the interval $[1, G_k(r))$ if and only if the range of $\log \sigma_{-r,k}$ is dense in the interval $[0, \log(G_k(r)))$. For this reason, we will henceforth focus on the ranges of the functions $\log \sigma_{-r,k}$ for various values of k and r.

Theorem 2.1. *Let $k \in \mathbb{N}$, and let $r \in (1, \infty)$. The range of $\log \sigma_{-r,k}$ is dense in the interval $[0, \log(G_k(r)))$ if and only if $f_k(m, r) \le \log(G_k(r))$ for all $m \in \mathbb{N}$.*

Proof. First, suppose that there exists some $m \in \mathbb{N}$ such that $f_k(m, r) > \log(G_k(r))$. Then

$$\log\left(1 + \frac{1}{p_m^r}\right) + \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) > \log\left(\prod_{i=1}^{\infty}\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right)\right)$$
$$= \sum_{i=1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right),$$

which means that

$$\log\left(1 + \frac{1}{p_m^r}\right) > \sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right).$$

Fix some $N \in S_k$, and let $N = \prod_{i=1}^{v} q_i^{\gamma_i}$ be the canonical prime factorization of N. Note that $\gamma_i \le k$ for all $i \in \{1, 2, \ldots, v\}$ because $N \in S_k$. If $p_s | N$ for some $s \in \{1, 2, \ldots, m\}$, then

$$\log \sigma_{-r,k}(N) \ge \log\left(1 + \frac{1}{p_s^r}\right) \ge \log\left(1 + \frac{1}{p_m^r}\right).$$

On the other hand, if $p_s \nmid N$ for all $s \in \{1, 2, \ldots, m\}$, then

$$\log \sigma_{-r,k}(N) = \log\left(\prod_{i=1}^{v} \sigma_{-r,k}(q_i^{\gamma_i})\right) = \log\left(\prod_{i=1}^{v}\left(\sum_{j=0}^{\gamma_i} \frac{1}{q_i^{jr}}\right)\right)$$
$$\le \log\left(\prod_{i=1}^{v}\left(\sum_{j=0}^{k} \frac{1}{q_i^{jr}}\right)\right) < \log\left(\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right)\right)$$
$$= \sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right).$$

Because N was arbitrary, this shows that there is no element of the range of $\log \sigma_{-r,k}$ in the interval $\left(\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right), \log\left(1+\frac{1}{p_m^r}\right)\right)$. Therefore, the range of $\log \sigma_{-r,k}$ is not dense in $[0, \log(G_k(r)))$.

Conversely, suppose that $f_k(m,r) \leq \log(G_k(r))$ for all $m \in \mathbb{N}$. This is equivalent to the statement that

$$\log\left(1+\frac{1}{p_m^r}\right) \leq \sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right)$$

for all $m \in \mathbb{N}$. Choose some arbitrary $x \in (0, \log(G_k(r)))$. We will construct a sequence in the following manner. First, let $C_0 = 0$. Now, for each positive integer l, let $C_l = C_{l-1} + \log\left(\sum_{j=0}^{\alpha_l} \frac{1}{p_l^{jr}}\right)$, where α_l is the largest nonnegative integer less than or equal to k such that $C_{l-1} + \log\left(\sum_{j=0}^{\alpha_l} \frac{1}{p_l^{jr}}\right) \leq x$. Also, for each $l \in \mathbb{N}$, let $D_l = \log\left(\sum_{j=0}^{k} \frac{1}{p_l^{jr}}\right) - \log\left(\sum_{j=0}^{\alpha_l} \frac{1}{p_l^{jr}}\right)$, and let $E_l = \sum_{i=1}^{l} D_i$. Note that

$$\begin{aligned}\lim_{l\to\infty}(C_l + E_l) &= \lim_{l\to\infty}\left(\sum_{i=1}^{l} \log\left(\sum_{j=0}^{\alpha_i} \frac{1}{p_i^{jr}}\right) + \sum_{i=1}^{l} D_i\right) \\ &= \lim_{l\to\infty}\sum_{i=1}^{l} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) = \log(G_k(r)).\end{aligned}$$

Now, because the sequence $(C_l)_{l=1}^{\infty}$ is bounded and monotonic, we know that there exists some real number γ such that $\lim_{l\to\infty} C_l = \gamma$. Note that, for each $l \in \mathbb{N}$, C_l is in the range of $\log \sigma_{-r,k}$ because

$$C_l = \sum_{i=1}^{l} \log\left(\sum_{j=0}^{\alpha_i} \frac{1}{p_i^{jr}}\right) = \log\left(\prod_{i=1}^{l} \sigma_{-r}(p_i^{\alpha_i})\right) = \log \sigma_{-r,k}\left(\prod_{i=1}^{l} p_i^{\alpha_i}\right).$$

Therefore, if we can show that $\gamma = x$, then we will know (because we chose x arbitrarily) that the range of $\log \sigma_{-r,k}$ is dense in $[0, \log(G_k(r)))$, which will complete the proof.

Because we defined the sequence $(C_l)_{l=1}^{\infty}$ so that $C_l \leq x$ for all $l \in \mathbb{N}$, we know that $\gamma \leq x$. Now, suppose $\gamma < x$. Then $\lim_{l\to\infty} E_l = \log(G_k(r)) - \gamma > \log(G_k(r)) - x$. This implies that there exists some positive integer L such that $E_l > \log(G_k(r)) - x$ for all $l \geq L$. Let m be the smallest positive integer that satisfies $E_m > \log(G_k(r)) - x$. First, suppose $D_m \leq x - C_m$ so that $x \geq C_m + D_m = C_{m-1} + \log\left(\sum_{j=0}^{k} \frac{1}{p_m^{jr}}\right)$. This implies, by the definition of α_m, that $\alpha_m = k$. Then $D_m = 0$. If $m > 1$, then $E_{m-1} = E_m > \log(G_k(r)) - x$, which contradicts the minimality of m. On the other hand, if $m = 1$, then we have $0 = D_m = E_m > \log(G_k(r)) - x$, which is also a contradiction. Thus, we conclude that $D_m > x - C_m$. Furthermore,

$$\begin{aligned}\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) &= \log(G_k(r)) - \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) \\ &= \log(G_k(r)) - E_m - C_m < x - C_m < D_m, \qquad (1)\end{aligned}$$

and we originally assumed that $\log\left(1 + \frac{1}{p_m^r}\right) \leq \sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right)$. This means that $\log\left(1 + \frac{1}{p_m^r}\right) < D_m = \log\left(\sum_{j=0}^{k} \frac{1}{p_m^{jr}}\right) - \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right)$, or, equivalently, $\log\left(1 + \frac{1}{p_m^r}\right) + \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right) < \log\left(\sum_{j=0}^{k} \frac{1}{p_m^{jr}}\right)$. If $\alpha_m > 0$, we have

$$\begin{aligned}\log\left(\left(1 + \frac{1}{p_m^r}\right)^2\right) \leq \log\left(1 + \frac{1}{p_m^r}\right) + \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right) < \log\left(\sum_{j=0}^{k} \frac{1}{p_m^{jr}}\right) \\ < \log\left(\sum_{j=0}^{\infty} \frac{1}{p_m^{jr}}\right) = \log\left(\frac{p_m^r}{p_m^r - 1}\right),\end{aligned}$$

so $\left(1 + \frac{1}{p_m^r}\right)^2 < \frac{p_m^r}{p_m^r - 1}$. We may write this as $1 + \frac{2}{p_m^r} + \frac{1}{p_m^{2r}} < 1 + \frac{1}{p_m^r - 1}$, so $2 < \frac{p_m^r}{p_m^r - 1} = 1 + \frac{1}{p_m^r - 1}$. As $p_m^r > 2$, this is a contradiction. Hence, $\alpha_m = 0$.

By the definitions of α_m and C_m, we see that $C_{m-1} + \log\left(1 + \frac{1}{p_m^r}\right) > x$ and that $C_m = C_{m-1}$. Therefore, $\log\left(1 + \frac{1}{p_m^r}\right) > x - C_{m-1} = x - C_m$.

However, recalling from (1) that $\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) < x - C_m$, we find that $\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) < \log\left(1 + \frac{1}{p_m^r}\right)$, which we originally assumed was false. Therefore, $\gamma = x$, so the proof is complete. $\square$

Given some positive integer k, we may use Theorem 2.1 to find the values of $r > 1$ such that the range of $\log \sigma_{-r,k}$ is dense in $[0, \log(G_k(r)))$. To do so, we only need to find the values of $r > 1$ such that $f_k(m, r) \leq \log(G_k(r))$ for all $m \in \mathbb{N}$. However, this is still a somewhat difficult problem. Luckily, we can make the problem much simpler with the use of the following theorem. We first need a quick lemma.

Lemma 2.1. *If $j \in \mathbb{N} \backslash \{1, 2, 4\}$, then* $\frac{p_{j+1}}{p_j} < \sqrt{2}$.

Proof. Pierre Dusart [2] has shown that, for $x \geq 396\,738$, there must be at least one prime in the interval $\left[x, x + \frac{x}{25 \log^2 x}\right]$. Therefore, whenever $p_j > 396\,738$, we may set $x = p_j + 1$ to get $p_{j+1} \leq (p_j + 1) + \frac{p_j + 1}{25 \log^2(p_j + 1)} < \sqrt{2} p_j$. Using Mathematica 9.0 [3], we may quickly search through all the primes less than 396 738 to conclude the desired result. $\square$

Remark 2.1. There is an identical statement and proof of Lemma 2.1 in [1], but we include it again here for the sake of completeness (and so that we may later refer to Lemma 2.1 with a name).

Theorem 2.2. *Let $k \in \mathbb{N}$, and let $r \in (1, 2]$. The range of the function $\log \sigma_{-r,k}$ is dense in the interval $[0, \log(G_k(r)))$ if and only if*
$f_k(m, r) \leq \log(G_k(r))$ for all $m \in \{1, 2, 4\}$.

Proof. In light of Theorem 2.1, we simply need to show that if
$f_k(m, r) \leq \log(G_k(r))$ for all $m \in \{1, 2, 4\}$, then $f_k(m, r) \leq \log(G_k(r))$ for all $m \in \mathbb{N}$. Thus, let us assume that k and r are such that $f_k(m, r) \leq \log(G_k(r))$ for all $m \in \{1, 2, 4\}$.

Now, if $m \in \mathbb{N}\backslash\{1,2,4\}$, then, by Lemma 2.1, $\dfrac{p_{m+1}}{p_m} < \sqrt{2} \le \sqrt[r]{2}$, which implies that $\dfrac{2}{p_{m+1}^r} > \dfrac{1}{p_m^r}$. We then have

$$\begin{aligned}
f_k(m+1,r) &= \log\left(1+\frac{1}{p_{m+1}^r}\right) + \sum_{i=1}^{m+1} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) \\
&\ge 2\log\left(1+\frac{1}{p_{m+1}^r}\right) + \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) \\
&> \log\left(1+\frac{2}{p_{m+1}^r}\right) + \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) \\
&> \log\left(1+\frac{1}{p_m^r}\right) + \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right) = f_k(m,r).
\end{aligned}$$

This means that $f_k(3,r) < f_k(4,r) \le \log(G_k(r))$. Furthermore, $f_k(m,r) < \log(G_k(r))$ for all $m \ge 5$ because $(f_k(m,r))_{m=5}^{\infty}$ is a strictly increasing sequence and $\lim\limits_{m\to\infty} f_k(m,r) = \log(G_k(r))$. $\square$

We now have a somewhat simple way to check whether or not the range of $\log\sigma_{-r,k}$ is dense in $[0, \log(G_k(r)))$ for given $k \in \mathbb{N}$ and $r \in (1,2]$, but we can do better. In what follows, we will let $T_k(m,r) = f_k(m,r) - \log(G_k(r))$.

Lemma 2.2. *For fixed $k \in \mathbb{N}$ and $m \in \{1,2,4\}$, $T_k(m,r)$ is a strictly increasing function in the variable r for all $r \in \left(1, \frac{7}{3}\right)$.*

Proof. $T_k(m,r) = \log\left(1+\dfrac{1}{p_m^r}\right) - \displaystyle\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right)$, so, for fixed $k \in \mathbb{N}$ and $m \in \{1,2,4\}$, we have

$$\frac{d}{dr} T_k(m,r) = \sum_{i=m+1}^{\infty} \left(\left(\frac{\sum_{a=1}^{k} a p_i^{-ar}}{\sum_{b=0}^{k} p_i^{-br}}\right) \log p_i\right) - \frac{\log p_m}{p_m^r + 1}.$$

Observe that, for any $p_i \in \mathbb{P}$, $k \in \mathbb{N}$, and $r \in \left(1, \dfrac{7}{3}\right)$, we have $\dfrac{\sum_{a=1}^{k} a p_i^{-ar}}{\sum_{b=0}^{k} p_i^{-br}} \geq \dfrac{p_i^{-r}}{1+p_i^{-r}} = \dfrac{1}{p_i^r+1}$. Therefore, in order to show that $\dfrac{d}{dr} T_k(m,r) > 0$, it suffices to show that $\displaystyle\sum_{i=m+1}^{\infty} \frac{\log p_i}{p_i^r+1} > \frac{\log p_m}{p_m^r+1}$.

For each $m \in \{1,2,4\}$, define the function $J_m \colon \left(1, \dfrac{7}{3}\right] \to \mathbb{R}$ by

$$J_m(x) = \frac{\log p_m}{p_m^x+1} - \sum_{i=m+1}^{m+6} \frac{\log p_i}{p_i^x+1}.$$

One may verify, for each $m \in \{1,2,4\}$, that the function J_m is increasing on the interval $\left(1, \dfrac{7}{3}\right)$ and that $J_m\left(\dfrac{7}{3}\right) < 0$. Thus, for $m \in \{1,2,4\}$,

$$\frac{\log p_m}{p_m^r+1} < \sum_{i=m+1}^{m+6} \frac{\log p_i}{p_i^r+1} < \sum_{i=m+1}^{\infty} \frac{\log p_i}{p_i^r+1}.$$

This completes the proof. □

Lemma 2.3. *For each positive integer k, the functions $T_k(1,r)$ and $T_k(2,r)$ each have precisely one root for $r \in (1,2]$.*

Proof. Fix some $k \in \mathbb{N}$. First, observe that $\displaystyle\lim_{r\to 1^+} T_k(1,r) = -\infty$ and $\displaystyle\lim_{r\to 1^+} T_k(2,r) = -\infty$. Also, when viewed as single-variable functions of r, $T_k(1,r)$ and $T_k(2,r)$ are continuous over the interval $(1,2]$. Therefore, if we invoke Lemma 2.2 and the Intermediate Value Theorem, we see that it is sufficient to show that $T_k(1,2)$ and $T_k(2,2)$ are positive. We have

$$T_k(1,2) = \log\left(1+\frac{1}{2^2}\right) - \sum_{i=2}^{\infty} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{2j}}\right) > \log\left(\frac{5}{4}\right) - \sum_{i=2}^{\infty} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_i^{2j}}\right)$$

$$= \log\left(\frac{5}{4}\right) - \log\left(\prod_{i=2}^{\infty} \frac{p_i^2}{p_i^2-1}\right) = \log\left(\frac{5}{4}\right) + \log\left(\frac{4}{3}\right) - \log(\zeta(2))$$

$$= \log\left(\frac{10}{\pi^2}\right) > 0$$

and

$$T_k(2,2) = \log\left(1+\frac{1}{3^2}\right) - \sum_{i=3}^{\infty}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{2j}}\right) > \log\left(\frac{10}{9}\right) - \sum_{i=3}^{\infty}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{2j}}\right)$$

$$= \log\left(\frac{10}{9}\right) - \log\left(\prod_{i=3}^{\infty}\frac{p_i^2}{p_i^2-1}\right) = \log\left(\frac{10}{9}\right) + \log\left(\frac{9}{8}\right) + \log\left(\frac{4}{3}\right) - \log(\zeta(2))$$

$$= \log\left(\frac{10}{\pi^2}\right) > 0.$$

□

Definition 2.2. For $k \in \mathbb{N}$ and $m \in \{1,2,4\}$, we define $R_k(m)$ by

$$R_k(m) = \begin{cases} r_0, & \text{if } T_k(m,r_0) = 0 \text{ and } 1 < r_0 < 2; \\ 2, & \text{if } T_k(m,r) < 0 \text{ for all } r \in (1,2). \end{cases}$$

Also, for each positive integer k, let M_k be the smallest element m of $\{1,2,4\}$ that satisfies $R_k(m) = \min(R_k(1), R_k(2), R_k(4))$.

Remark 2.2. Observe that, for each $k \in \mathbb{N}$, Lemma 2.2, when combined with the fact that $\lim_{r\to 1^+} T_k(m,r) = -\infty$ for all $m \in \{1,2,4\}$, guarantees that the function R_k is well-defined. Furthermore, note that Lemma 2.3 tells us that $R_k(M_k) < 2$. Essentially, M_k is the element m of the set $\{1,2,4\}$ that gives $g(r) = T_k(m,r)$ the smallest root in the interval $(1,2)$, and if multiple values of m give $g(r)$ this minimal root, M_k is simply defined to be the smallest such m.

Lemma 2.4. *For all $k \in \mathbb{N}$ and $m \in \{1,2,4\}$, $R_{k+1}(m) \geq R_k(m)$, where equality holds if and only if $m = 4$ and $R_k(m) = 2$.*

Proof. Fix $k \in \mathbb{N}$ and $m \in \{1,2,4\}$. Note that if $f_k(m,r) \leq \log(G_k(r))$ for some $r \in (1,2]$, then

$$f_{k+1}(m,r) - \sum_{i=1}^{m}\log\left(\sum_{j=0}^{k+1}\frac{1}{p_i^{jr}}\right) = \log\left(1+\frac{1}{p_m^r}\right)$$

$$= f_k(m,r) - \sum_{i=1}^{m}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right) \leq \log(G_k(r)) - \sum_{i=1}^{m}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right)$$

$$= \sum_{i=m+1}^{\infty}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right) < \sum_{i=m+1}^{\infty}\log\left(\sum_{j=0}^{k+1}\frac{1}{p_i^{jr}}\right)$$

$$= \log(G_{k+1}(r)) - \sum_{i=1}^{m} \log\left(\sum_{j=0}^{k+1} \frac{1}{p_i^{jr}}\right),$$

so $f_{k+1}(m, r) < \log(G_{k+1}(r))$. We now consider two cases.

Case 1: $T_k(m, r_0) = 0$ for some $r_0 \in (1, 2)$. In this case, $R_k(m) = r_0$, so $T_k(m, R_k(m)) = 0$. Therefore, $f_k(m, R_k(m)) = \log(G_k(R_k(m)))$. By the argument made in the preceding paragraph, we conclude that $f_{k+1}(m, R_k(m)) < \log(G_{k+1}(R_k(m)))$, which is equivalent to the statement $T_{k+1}(m, R_k(m)) < 0$. Either $R_{k+1}(m) = 2 > R_k(m)$ or $T_{k+1}(m, R_{k+1}(m)) = 0 > T_{k+1}(m, R_k(m))$. In the latter case, Lemma 2.2 tells us that $R_{k+1}(m) > R_k(m)$.

Case 2: $T_k(m, r) < 0$ for all $r \in (1, 2)$. In this case, $R_k(m) = 2$, and $f_k(m, 2) \leq \log(G_k(2))$. By the argument made in the beginning of this proof, we conclude that $f_{k+1}(m, 2) < \log(G_{k+1}(2))$. Therefore, combining Lemma 2.2 and Definition 2.2, we may conclude that $R_{k+1}(m) = R_k(m) = 2$. Note that, by Lemma 2.3, this case can only occur if $m = 4$. □

We now mention some numerical results, obtained using Mathematica 9.0, that we will use to prove our final lemma and theorem.

Let us define a function $V_k(m, r)$ by $V_k(m, r) = \log\left(1 + \frac{1}{p_m^r}\right) - \sum_{i=m+1}^{10^5} \log\left(\sum_{j=0}^{k} \frac{1}{p_i^{jr}}\right)$. Then, for fixed $k \in \mathbb{N}$ and $m \in \{1, 2, 4\}$, we have

$$\frac{d}{dr} V_k(m, r) = \sum_{i=m+1}^{10^5} \left(\left(\frac{\sum_{a=1}^{k} a p_i^{-ar}}{\sum_{b=0}^{k} p_i^{-br}}\right) \log p_i\right) - \frac{\log p_m}{p_m^r + 1}$$

$$> \sum_{i=m+1}^{m+6} \left(\frac{\log p_i}{p_i^r + 1}\right) - \frac{\log p_m}{p_m^r + 1}.$$

Referring to the last two sentences of the proof of Lemma 2.2, we see that $\frac{d}{dr} V_k(m, r) > 0$ for $r \in \left(1, \frac{7}{3}\right)$ when $k \in \mathbb{N}$ and $m \in \{1, 2, 4\}$ are fixed. In particular, we will make use of the fact that $V_1(1, r)$ is an increasing function of r on the interval $\left(1, \frac{7}{3}\right)$. We may easily verify that $V_1(1, 1) < 0 < V_1\left(1, \frac{7}{3}\right)$, so there exists a unique number $r_1 \in \left(1, \frac{7}{3}\right)$ such that $V_1(1, r_1) = 0$. Mathematica approximates this value as $r_1 \approx 1.864633$. We have

$$V_1(1, r_1) = 0 = T_1(1, R_1(1)) = \log\left(1 + \frac{1}{2^{R_1(1)}}\right) - \sum_{i=2}^{\infty} \log\left(1 + \frac{1}{p_i^{R_1(1)}}\right)$$

$$< \log\left(1 + \frac{1}{2^{R_1(1)}}\right) - \sum_{i=2}^{10^5} \log\left(1 + \frac{1}{p_i^{R_1(1)}}\right) = V_1(1, R_1(1)).$$

Because $V_1(1, r)$ is increasing, we find that $R_1(1) > r_1$. The important point here is that $R_1(1) \in (1.8638, 2)$. One may confirm, using a simple graphing calculator, that $\left(1 + \frac{1}{2^r}\right)\left(\frac{3^r}{3^r + 1}\right) > 1 + \frac{1}{3^r}$ for all $r \in (1.8638, 2)$. Therefore, we may write

$$T_1(2, R_1(2)) = 0 = T_1(1, R_1(1)) = \log\left(1 + \frac{1}{2^{R_1(1)}}\right) - \sum_{i=2}^{\infty} \log\left(1 + \frac{1}{p_i^{R_1(1)}}\right)$$

$$= \log\left(\left(1 + \frac{1}{2^{R_1(1)}}\right)\left(\frac{3^{R_1(1)}}{3^{R_1(1)} + 1}\right)\right) - \sum_{i=3}^{\infty} \log\left(1 + \frac{1}{p_i^{R_1(1)}}\right)$$

$$> \log\left(1 + \frac{1}{3^{R_1(1)}}\right) - \sum_{i=3}^{\infty} \log\left(1 + \frac{1}{p_i^{R_1(1)}}\right) = T_1(2, R_1(1)).$$

As $T_1(2, r)$ is increasing on the interval $(1, 2)$ (by Lemma 2.2), we find that $R_1(2) > R_1(1)$. We may use a similar argument, invoking the fact that

$$\left(1 + \frac{1}{2^r}\right)\left(\frac{3^r}{3^r + 1}\right)\left(\frac{5^r}{5^r + 1}\right)\left(\frac{7^r}{7^r + 1}\right) > 1 + \frac{1}{7^r}$$

for all $r \in (1.8638, 2)$, to show that $R_1(4) > R_1(1)$. Thus, $R_1(1) = \min(R_1(1), R_1(2), R_1(4))$, so $M_1 = 1$.

Now, one may easily verify that, for all $r \in (1.67, 1.98)$,

$$1 + \frac{1}{2^r} < \left(1 + \frac{1}{3^r}\right)\left(1 + \frac{1}{3^r} + \frac{1}{3^{2r}}\right) \tag{2}$$

and

$$1 + \frac{1}{3^r} > \left(\frac{5^r}{5^r - 1}\right)\left(\frac{7^r + 1}{7^r - 1}\right). \tag{3}$$

If we fix some integer $k \geq 2$, then, for all $r \in (1.67, 1.98)$, we may use (2) to write

$$f_k(1,r) = \log\left(1+\frac{1}{2^r}\right) + \log\left(\sum_{j=0}^{k}\frac{1}{2^{jr}}\right)$$

$$< \log\left(\left(1+\frac{1}{3^r}\right)\left(1+\frac{1}{3^r}+\frac{1}{3^{2r}}\right)\right) + \log\left(\sum_{j=0}^{k}\frac{1}{2^{jr}}\right)$$

$$\leq \log\left(1+\frac{1}{3^r}\right) + \log\left(\sum_{j=0}^{k}\frac{1}{3^{jr}}\right) + \log\left(\sum_{j=0}^{k}\frac{1}{2^{jr}}\right) = f_k(2,r).$$

Similarly, for all $r \in (1.67, 1.98)$, we may use (3) to write

$$f_k(2,r) = \log\left(1+\frac{1}{3^r}\right) + \sum_{i=1}^{2}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right)$$

$$> \log\left(\left(\frac{5^r}{5^r-1}\right)\left(\frac{7^r+1}{7^r-1}\right)\right) + \sum_{i=1}^{2}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right)$$

$$= \log\left(\sum_{j=0}^{\infty}\frac{1}{5^{jr}}\right) + \log\left(\sum_{j=0}^{\infty}\frac{1}{7^{jr}}\right) + \log\left(1+\frac{1}{7^r}\right) + \sum_{i=1}^{2}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right)$$

$$> \log\left(\sum_{j=0}^{k}\frac{1}{5^{jr}}\right) + \log\left(\sum_{j=0}^{k}\frac{1}{7^{jr}}\right) + \log\left(1+\frac{1}{7^r}\right) + \sum_{i=1}^{2}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right)$$

$$= \log\left(1+\frac{1}{7^r}\right) + \sum_{i=1}^{4}\log\left(\sum_{j=0}^{k}\frac{1}{p_i^{jr}}\right) = f_k(4,r).$$

We now know that $f_k(2,r) > f_k(1,r), f_k(4,r)$ whenever $k \in \mathbb{N}\backslash\{1\}$ and $r \in (1.67, 1.98)$. As our last preliminary computation, we need to evaluate $\lim\limits_{n\to\infty} R_n(2)$. For each positive integer n, $R_n(2)$ is the unique solution $r \in (1,2)$ of the equation $f_n(2,r) = \log(G_n(r))$. We may rewrite this equation as

$$\log\left(1+\frac{1}{3^r}\right) = \sum_{i=3}^{\infty}\log\left(\sum_{j=0}^{n}\frac{1}{p_i^{jr}}\right),$$

or, equivalently, $\left(\sum_{j=0}^{n} \frac{1}{2^{jr}}\right)\left(\sum_{j=0}^{n} \frac{1}{3^{jr}}\right)\left(1+\frac{1}{3^r}\right) = \prod_{i=1}^{\infty}\left(\sum_{j=0}^{n} \frac{1}{p_i^{jr}}\right)$. Because the summations and the product in this equation converge (for $r > 1$) as $n \to \infty$, we see that $\lim_{n\to\infty} R_n(2)$ is simply the solution (in the interval $(1,2)$) of the equation $\lim_{n\to\infty}\left[\left(\sum_{j=0}^{n} \frac{1}{2^{jr}}\right)\left(\sum_{j=0}^{n} \frac{1}{3^{jr}}\right)\left(1+\frac{1}{3^r}\right)\right] = \lim_{n\to\infty}\left[\prod_{i=1}^{\infty}\left(\sum_{j=0}^{n} \frac{1}{p_i^{jr}}\right)\right]$, which we may write as

$$\left(\frac{2^r}{2^r-1}\right)\left(\frac{3^r+1}{3^r-1}\right) = \zeta(r). \tag{4}$$

The only solution to this equation in the interval $(1,2)$ is $r = \eta \approx 1.8877909$ [1]. For now, the important piece of information to note is that $\lim_{n\to\infty} R_n(2) \in (1.67, 1.98)$.

Lemma 2.5. *For all integers $k > 1$, $M_k = 2$.*

Proof. Fix some integer $k > 1$. First, suppose $M_k = 1$. This means that $R_k(1) \leq R_k(2)$. Using Lemma 2.4 and the facts that $R_1(1) > 1.8638$ and $\lim_{n\to\infty} R_n(2) < 1.98$, we have

$$1.8638 < R_1(1) < R_k(1) \leq R_k(2) < \lim_{n\to\infty} R_n(2) < 1.98.$$

Therefore, $R_k(1) \in (1.67, 1.98)$, so we know that $f_k(2, R_k(1)) > f_k(1, R_k(1)) = \log(G_k(R_k(1)))$. Hence, $T_k(2, R_k(1)) > 0$. Lemma 2.2, when coupled with our assumption that $R_k(1) \leq R_k(2)$, then implies that $T_k(2, R_k(2)) > 0$. However, this is impossible because Lemma 2.3 and the definition of $R_k(2)$ guarantee that $T_k(2, R_k(2)) = 0$.

Next, suppose $M_k = 4$. This means that $R_k(4) < R_k(2)$. Also, referring to Remark 2.2, we see that $R_k(4) < 2$. Therefore, by the definition of $R_k(4)$, we find that $f_k(4, R_k(4)) = \log(G_k(R_k(4)))$. Now, we may write

$$1.8638 < R_1(1) < R_1(4) < R_k(4) < R_k(2) < \lim_{n\to\infty} R_n(2) < 1.98.$$

As $R_k(4) \in (1.67, 1.98)$, we have

$$f_k(2, R_k(4)) > f_k(4, R_k(4)) = \log(G_k(R_k(4))).$$

Thus, $T_k(2, R_k(4)) > 0$. Using Lemma 2.2 and our assumption that $R_k(4) < R_k(2)$, we get $T_k(2, R_k(2)) > 0$. Again, this is a contradiction. □

We now culminate our work with a final definition and theorem.

Definition 2.3. Let η_1 be the unique real number in the interval $(1,2)$ that satisfies
$$\left(1+\frac{1}{2^{\eta_1}}\right)^2=\frac{\zeta(\eta_1)}{\zeta(2\eta_1)}.$$
For each integer $k>1$, let η_k be the unique real number in the interval $(1,2)$ that satisfies
$$\left(\sum_{j=0}^{k}\frac{1}{2^{\eta_k}}\right)\left(\sum_{j=0}^{k}\frac{1}{3^{\eta_k}}\right)\left(1+\frac{1}{3^{\eta_k}}\right)=\frac{\zeta(\eta_k)}{\zeta((k+1)\eta_k)}.$$

Remark 2.3. Using Definition 2.1 to manipulate the equation $f_k(M_k,R_k(M_k))=\log(G_k(R_k(M_k)))$ and using the fact that
$$M_k=\begin{cases}1, & \text{if } k=1;\\ 2, & \text{if } k\in\mathbb{N}\backslash\{1\},\end{cases}$$
one can see that η_k is simply $R_k(M_k)$. Furthermore, Lemma 2.2 tells us that, for each positive integer k, the value of η_k is, in fact, unique.

Theorem 2.3. *Let k be a positive integer. If $r>1$, then the range of the function $\sigma_{-r,k}$ is dense in the interval $\left[1,\dfrac{\zeta(r)}{\zeta((k+1)r)}\right)$ if and only if $r\le\eta_k$.*

Proof. Let k be a positive integer, and let $r\in\left(1,\frac{7}{3}\right)$. Suppose $r\le\eta_k$. Then, by the definition of M_k and the fact that $\eta_k=R_k(M_k)$, we see that $r\le R_k(m)$ for all $m\in\{1,2,4\}$. Lemma 2.2 then guarantees that $T_k(m,r)\le 0$ for all $m\in\{1,2,4\}$, which means that $f_k(m,r)\le\log(G_k(r))$ for all $m\in\{1,2,4\}$. Theorem 2.2 then tells us that the range of $\log\sigma_{-r,k}$ is dense in the interval $[0,\log(G_k(r)))$, which implies that the range of $\sigma_{-r,k}$ is dense in $[1,G_k(r))$. Now, suppose that $r>\eta_k$. Then $T_k(M_k,r)>T_k(M_k,R_k(M_k))=0$, so, $f_k(M_k,r)>\log(G_k(r))$. This means that the range of $\log\sigma_{-r,k}$ is not dense in $[0,\log(G_k(r)))$, which is equivalent to the statement that the range of $\sigma_{-r,k}$ is not dense in $[1,G_k(r))$.

We now need to show that, for any $k\in\mathbb{N}$, the range of $\sigma_{-r,k}$ is not dense in $[0,\log(G_k(r)))$ for all $r>\frac{7}{3}$. To do so, it suffices to show that $f_k(1,r)>\log(G_k(r))$ for all $r>\frac{7}{3}$, which means that we only need to show that $\left(1+\frac{1}{2^r}\right)\sum_{j=0}^{k}\frac{1}{2^{jr}}>G_k(r)$ for $r>\frac{7}{3}$. Now, because $G_k(r)<\zeta(r)$, we see that it suffices to show that $\left(1+\frac{1}{2^r}\right)^2 > \zeta(r)$ for $r > \frac{7}{3}$. One may easily verify that this inequality holds for $\frac{7}{3} < r \leq 3$. For $r > 3$, we have

$$\left(1+\frac{1}{2^r}\right)^2 > 1+\frac{1}{2^r}+\frac{1}{2}\left(\frac{1}{2^{r-1}}\right) > 1+\frac{1}{2^r}+\frac{1}{(r-1)2^{r-1}}$$
$$= 1+\frac{1}{2^r}+\int_2^\infty \frac{1}{x^r}dx > \zeta(r).$$

□

3. An Open Problem

As the author has done for a density problem related to generalizations divisor functions without restricted domains [1], we pose a question related to the number of "gaps" in the range of $\sigma_{-r,k}$ for various k and r. That is, given positive integers k and L, what are the values of $r > 1$ such that the closure of the range of $\sigma_{-r,k}$ is a union of exactly L disjoint subintervals of $\left[1, \frac{\zeta(r)}{\zeta((k+1)r)}\right]$?

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