Uniform Distribution Theory 9 (2014), no.2, 125–134

uniform distribution theory

A REMARK ON THE DISTRIBUTION OF CHEBYCHEV POLYNOMIALS ON $[-1,1]$

Kwo Chan — Radhakrishnan Nair

ABSTRACT. Let $(T_n(x))_{n\geq 0}$ denote the sequences of Chebychev polynomials of the first kind defined by the recursive relation $T_0(x)\equiv 1$, $T_1(x)=x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. We observe that the sequences $\big(T_n(x)\big)_{n\geq 0}$ is asymptotically distributed with respect to the measure with the density $\pi^{-1}(1-x^2)^{-\frac{1}{2}}$ on $[-1,1]$ for almost all x. Various refinements of this observation are also noted.

Communicated by O. Strauch

1. Introduction

Recall that a sequence of real numbers $(x_n)_{n\geq 0}$ is uniformly distributed modulo one if for each interval I contained in $[0,1)$, that is closed on the left and open on the right,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\chi_I(\{x_n\}) \;=\; |I|\,.$$

Here χ_I denotes the characteristic function of the interval I and $|I|$ denotes its length. We have also used $\{y\}$ to denote the fractional part of a real number y. For a sequence of real numbers $x_0,\dots,x_{N-1}$, let

$$D(x_0,\dots,x_{N-1}) = \sup_{I\subseteq[0,1)}\left|\frac{1}{N}\sum_{n=0}^{N-1}\chi_I(\{x_n\}) - |I|\right| \quad (n=0,1,\dots),$$

where the supremum is taken over all intervals I, closed on the left and open on the right. We call $D(x_0,\dots,x_{N-1})$ the *discrepancy* of the numbers $x_0,\dots,x_{N-1}$. The discrepancy of a sequence of numbers $(x_n)_{n\geq 0}$ tends to zero as N tends to infinity if and only if the sequence is uniformly distributed modulo one. This means that for a uniformly distributed sequence of real numbers $(x_n)_{n\geq 0}$,

2010 Mathematics Subject Classification: 11K35, 40A05, 37A05.
Keywords: Chebychev polynomials, asymptotic distribution.

the rate for decay of $D(x_0, \ldots x_{N-1})$ as N tends to infinity, provides a measure of the degree of uniformity of distribution. A classical theorem of H. Weyl [We] tells us that if $(a_n)_{n\geq 0}$ is a sequence of distinct integers then $(a_n x)_{n\geq 0}$ is uniformly distributed modulo one for a.e. x. As usual if a property holds except for a set of Lebesgue measure zero, it is said that holds almost everywhere, abbreviated '*a.e.*'. Let $D(N,x) = D(\{a_0x\}, \cdots, \{a_{N-1}x\})$ $(N = 1, 2, \ldots)$. Given $\epsilon > 0$ the following are known:

1. [AMZ]**:** $D(N,x) = o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}}(\log\log N)^{-\frac{1}{2}+\epsilon})$ a.e.
2. [P1]**:** If there exists $q > 1$ such that $a_{k+1} \geq q a_k$ for $k \geq 1$, then there exists a constant $C_1 > 0$ such that
$$D(N,x) \leq C_1 N^{-\frac{1}{2}}(\log\log N)^{-\frac{1}{2}+\epsilon} \ a.e.$$
3. [P2]**:** If $(a_n)_{n\geq 0}$ is generated multiplicatively by a finite set of co-prime integers, then there exists a constant $C_2 > 0$ such that
$$D(N,x) \leq C_2 N^{-\frac{1}{2}}(\log\log N)^{-\frac{1}{2}+\epsilon} \ a.e.$$
4. [Po], [dM]**:** If there exists $q > 1$ such that $a_{k+1} \geq q a_k$ for $k \geq 1$, then the set of x such that the sequence $(\{a_n x\})_{n\geq 0}$ is not dense, has Hausdorff dimension 1.
5. [Be]**:** If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the set of points x, where $(\{a_n x\})_{n\geq 0}$ is not dense, has Hausdorff dimension 0.
6. [Ba]**:** If $a_n = O(n^p)$ as n tends to infinity, for $p > 1$ and $q \in (0, \frac{1}{2})$, then the set
$$E_q = \{x : \limsup_{N\to\infty} N^q D(N,x) > 0\}$$
has Hausdorff dimension bounded above by $1 - \frac{1-2q}{p+q}$.

In this paper we give some applications of these estimates and ergodic theory to study the distribution of Chebychev polynomials.

2. Chebychev Polynomials

Let $\big(T_n(x)\big)_{n\geq 0}$ denote the sequence of Chebychev polynomials of the first kind defined by the recursive relation $T_0(x) \equiv 1$, $T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ $(n \geq 1)$. We can also define Chebychev polynomials of the second kind as $U_0(x) \equiv 1$, $U_1(x) = 2x$ and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}$ $(n \geq 1)$, but these will not figure in our considerations. These polynomials are defined on the whole real line and more broadly if you wish to but we shall confine attention to the interval $[-1,1]$. This is interesting set is because for each $n \geq 1$, the polynomial $T_n(x)$ maps $[-1,1]$ to itself. In effect $[-1,1]$ is an attractor for the dynamical system $(T_n^k(x))_{k\geq 0}$ in the complex plane. One can readily check that $T_n(x)$ can be defined for $x \in [-1,1]$ by $T_n(x) = \cos\bigl(n \arccos(x)\bigr)$. From this one obtains the semi-group property $T_n\bigl(T_m(x)\bigr) = T_{nm}(x)$ for $n, m \geq 1$ on $[-1,1]$. To envision the action of a Chebychev polynomial T_n on a point $x \in [-1,1]$ imagine the unit disc lying in the complex plane containing the interval $[-1,1]$. To a point $x \in [-1,1]$ associate the point directly above it on the unit circle S^1 and suppose θ is the angle subtended by the ray from the origin to this point on the circle. To this point on the unit circle which we shall also denote by θ we apply the map $M_n : S^1 \to S^1$ by $\theta \to n\theta \mod 2\pi$. Now we note that $T_n(x)$ coincides with the projection of $n\theta \mod 2\pi$ onto $[-1,1]$. This way of describing $T_n(x)$ is well defined even if we start by projecting downward instead of projecting upward. What this description makes clear is that the distribution of $\bigl(T_n(x)\bigr)_{n\geq 0}$ is intimately linked to the distribution of $(n\theta)_{n\geq 0}$ on S^1. Consider the measure defined for Lebesgue measurable $A \subseteq [-1,1]$ by

$$\mu(A) = \frac{1}{\pi} \int_A \frac{dt}{\sqrt{1-x^2}} \,.$$

This measure μ [1] assigns to each $A \subseteq (-1,1)$ the measure of the subset of S^1 that project vertically to A. Suppose A^1 and A^2 denote the subsets of S^1 that project downward and upward onto A respectively. Thus for any sequence of natural numbers $(a_k)_{k\geq 0}$, if λ denotes arc length measure on S^1,

$$\left| \frac{1}{N} \sum_{k=1}^{N} \chi_A\bigl(T_{a_k}(x)\bigr) - \mu(A) \right| \leq 2 \sum_{i=1}^{2} \left| \frac{1}{N} \sum_{k=1}^{N} \chi_{A^i}(a_k\theta) - \lambda(A^i) \right| . \tag{2.1}$$

Let

$$D_\mu(N,x) = \sup_{I \subseteq [-1,1]} \left| \frac{1}{N} \sum_{k=1}^{N} \chi_I\bigl(T_{a_k}(x)\bigr) - \mu(I) \right| ,$$

where the supremum is taken over all internals $I \subseteq [-1,1]$. We have proved the following theorem which follows from the upper bound (2.1) and the discrepancy estimates 1–6.

[1] The density $\frac{1}{\pi\sqrt{1-x^2}}$ is known as the arcsin density and has made several appearances in probability theory. See [SZ] for a characterization.

Theorem 2.1. *For a sequence of integers $(a_k)_{k\geq 1}$ and given $\epsilon > 0$ the following are true:*

(i) *We have $D_\mu(N,x) = o(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2}}(\log\log N)^{-\frac{1}{2}+\epsilon})$ a.e.*

(ii) *If there exists $q > 1$ such that $a_{k+1} \geq qa_k$ for $k \geq 1$, then there exist a constant C_3 such that*

$$D_\mu(N,x) \leq C_3 N^{-\frac{1}{2}}(\log\log N)^{-\frac{1}{2}+\epsilon} \ \text{a.e.}$$

(iii) *If $(a_n)_{n\geq 1}$ is generated multiplicatively by a finite set of co-prime integers, then there exist a constants $C_4 > 0$ such that*

$$D_\mu(N,x) \leq C_4 N^{-\frac{1}{2}}(\log\log N)^{-\frac{1}{2}+\epsilon} \ \text{a.e.}$$

(iv) *If there exists $q > 1$ such that $a_{k+1} \geq qa_k$ for all $k \geq 1$, then the set of x such that the sequence $\big(T_{a_n}(x)\big)_{n\geq 0}$ is not dense has Hausdorff dimension 1.*

(v) *If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the set of points x, where $\big(T_{a_n}(x)\big)_{n\geq 0}$ is not dense, has Hausdorff dimension 0.*

(vi) *If $a_n = O(n^p)$ as n tends to infinity, for $p > 1$ and $q \in (0, \frac{1}{2})$, then the set*

$$F_q = \{x : \limsup_{N\to\infty} N^q D_\mu(N,x) > 0\}$$

has Hausdorff dimension bounded above by $1 - \frac{1-2q}{p+q}$.

Let $P(x)$ denote a polynomial that maps the natural numbers to themselves. By the Lagrange interpolation theorem this must have rational coefficients. We know for angle θ if $\frac{\theta}{2\pi}$ is irrational, that if $(p_n)_{n=1}^\infty$ denotes the sequence of rational primes ordered by absolute value, the sequences $\big(M_{P(n)}(\theta)\big)_{n=1}^\infty$ and $\big(M_{P(p_n)}(\theta)\big)_{n=1}^\infty$ are uniformly distributed on S^1. See [We] and [R], respectively. The following theorem is immediate.

Theorem 2.2. *Suppose $x \in [-1,1]$ is the projection of $\theta \in S^1$ for irrational $\frac{\theta}{2\pi}$. Then both the sequences $\big(T_{P(n)}(x)\big)_{n\geq 1}$ and $\big(T_{P(p_n)}(x)\big)_{n\geq 1}$ are asymptotically distribued on $[-1,1]$ with respect to μ.*

In [N3] it is shown that if $(a_n)_{n\geq 1}$ is either $\big(\{P(n)\}\big)_{n\geq 1}$ or $\big(\{P(p_n)\}\big)_{n\geq 1}$ as in Theorem 2.2, if $f \in L^p\big([0,1)\big)$ for $p > 1$ and $b > 1$ is an integer, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{b^{a_n}\theta\}) = \int_0^1 f(t)dt \ \text{a.e. in } \theta.$$

Arguing similarly to the proof of Theorem 2.1 we see this implies the following

Theorem 2.3. *Suppose $(a_n)_{n\geq 1}$ is as in Theorem 2.2. If $f \in L^p([-1,1])$ for $p > 1$ and $b > 1$ is an integer, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(T_{b^{a_n}}(x)\big) = \int_{-1}^{1} f(t)\frac{dt}{\pi\sqrt{1-t^2}}, \text{ a.e. in } x.$$

Using a general pointwise ergodic theorem of T. Bewley [Be] and an idea of J. M. Marstrand [M] we can prove the following theorem.

Theorem 2.4. *Suppose U is a subset of the natural numbers and that f is a μ integrable functions on $[-1,1]$. Let $(m_k)_{k\geq 1}$ denote the sequence of integers generated multiplicatively by the set U ordered by absolute value. We have two conclusions:*

(a) *If $(m_k)_{k\geq 1}$ is contained in a subset of the natural numbers generated multiplicatively by a finite set, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f\big(T_{m_k}(x)\big) = \int_{-1}^{1} f(t)\frac{dt}{\pi\sqrt{1-t^2}}, \quad \text{a.e. in } x. \tag{2.2}$$

(b) *If $(m_k)_{\geq 1}$ is not contained in a finitely generated subset of the natural numbers, then there is a G_δ set $B \subset [-1,1]$ such that the averages in (2.2) fail to converge almost everywhere for $f(x) = \chi_B(x)$.*

3. Proof of Theorem 2.4

We begin by describing a general framework. Let S be a countable abelian semigroup acting in a measure preserving fashion on a measure space $(\Omega, \mathcal{A}, \mu)$. That is to each $g \in S$ there exists a measurable map T_g of Ω such that

$$T_{g_1+g_2} = T_{g_1}(T_{g_2})$$

and for each A in the σ-algebra $\mathcal{A}$ we have

$$T_g^{-1}A = \{x \in \Omega : T_g x \in A\} \in \mathcal{A} \quad \text{for all} \quad g \in S \quad \text{with } \mu(T_g^{-1}A) = \mu(A).$$

Also let $(A_k)_{k=1}^{\infty}$ be a collection of subsets of S such that the following conditions hold:

(B1) $0 < \#A_k < \infty$.

(B2) $A_k \subset A_{k+1}$ for each $k \in \mathbf{N}$.

(B3) there exists a constant $M_2 > 0$ such that

$$\#\{A_k - A_k\} \le M_2 \# A_k \quad \text{for all } k \text{ in } \mathbf{N}.$$

(B4) (amenability)

$$\lim_{k\to\infty} \frac{\#\{(g + A_k) \triangle A_k\}}{\# A_k} = 0 \quad \text{for each } g \text{ in } S,$$

where we have used $A \triangle B$ to denote the symmetric difference of the sets A and B.

Here for a finite set A we have used $\#A$ to denote its cardinality, and $A - A$ to denote $\{x \in S : y + x \in A \text{ for some } y \in A\}$.

From the above data we construct, the averages

$$\pi_k(f)(x) = \frac{1}{\# A_k} \sum_{s\in A_k} f(T_s x) \quad \text{for} \quad k \in \mathbf{N},$$

where $f \in L^1(\Omega, \mathcal{A}, \mu)$. We have a special case of T. Bewley's theorem [Be].

Theorem 3.1. *Suppose* (B1), (B2), (B3) and (B4) are true. Then

$$\pi(f)(x) = \lim_{k\to\infty} \pi_k(f)(x)$$

exists μ a.e., with

$$\int_\Omega \pi(f) d\mu = \int_\Omega f d\mu \quad \text{and} \quad \big(\pi(f)\big)\big(T_s(x)\big) = \pi(f)(x)$$

for almost all x, for each $s \in S$.

We further specialise this theorem as follows. Let $S = \prod_{n\ge1} \mathbb{N}$, i.e., the direct product of the natural numbers with themselves countably many times. The set S may also be described as the space of sequences on elements of $\mathbb{N}$, all but finitely many of whose elements are non zero. We set

$$A_k = \{n_1^{s_1} \cdots n_l^{s_l} \cdots \le k : s = (s_i)_{i\ge1} \in S; (n_i)_{i\ge1} \subset U\}.$$

Clearly all the indefinite products $n_1^{s_1} \ldots n_l^{s_l} \ldots$ are actually finite and correspond to individual natural numbers in the sequence $(m_l)_{l\ge1}$. It is immediate that (B1), (B2) and (B3) are satisfied by $(A_k)_{k\ge1}$. Proving (B4) amounts to showing for each fixed $g \in S$ that

$$\#\{(g + A_k)\backslash A_k\}, \#\{A_k\backslash(g + A_k)\} = o(\# A_k),$$

as k tends to infinity. Clearly this involves getting estimates for the numbers $\#\{(g + A_k)\backslash A_k\}, \#\{A_k\backslash(g + A_k)\}$ and $\#A_k$. The authors know two methods for doing this. The first is the geometric approach of comparing the number of lattice points we are counting to the volumes of the regions they are in and estimating these volumes. The second is a inductive counting argument based on the number of generators at issue. In the case of $\mathbb{N}^r$ for finite $r \geq 1$ instead of S, these estimates are carried out using the first method in [N2] and the second in [N2]. Extending these arguments to S, in the case where $(m_k)_{k\geq 1}$ is contained in a finitely generated semigroup is a very routine exercise using the fundamental theorem of arithmetic and so we forgo the details. The property (B4) follows. Now take $\Omega = [-1, 1]$, $\mathcal{A}$ to be the Lebesgue σ-algebra and μ to be arc length measure on S^1 projected onto $[-1, 1]$. We also take $(T_g)_{g\in S}$ to be Chebychev polynomials indexed by the elements of the the sequence $(m_k)_{\geq 1}$. As noted earlier Chebychev polynomials preserve the measure μ and compose as a semigroup. Now note that the average $\pi_s(f)$ coincides with the average on the left hand side of (2.2). Theorem 3.1 tells us these averages converge to a limit pointwise that is invariant under the action of Chebychev polynomials. We have only to identify the limit. The maps M_n $(n \geq 1)$ are known to be ergodic. It is an elementary exercise to show that this imples the ergodicty and mixing of the Chebychev map T_n, using the fact that the maps are 'conjugate'. See [Wa, p. 59] for details.

It is also the case that a measurable function that is invariant under a μ measure preserving transformation which is also ergodic must be constant, see [Wa, p. 28]. It is immediate that the constant must be integral of the function. Thus Theorem 2.4 (a) is proved.

We now turn to the proof of Theorem 2.4 (b). Our main tool is the following [M].

Lemma 3.2. *Let $(m_k)_{k\geq 1}$ denote a strictly sequence of integers. Suppose for each pair of integers $q, v > 1$ that there exist pairs of sets of integers G, H such that*

(1) *$\#G > v\#H$,*

(2) *for every $g \in G$ there exists $\eta \geq 1$ such that $gm_k^{-1} \in H$ for all $k \in [\eta, \eta q]$.*

Then there exists a G_δ set B such that if $f = \chi_B$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\{m_k\theta\}),$$

fails to exist almost everywhere with respect to Lebesgue measure.

Using this lemma it is possible to prove that any sequence of natural numbers that is multiplicatively generated but not contained in a set of integers multiplicatively generated by any finite set must have the properties claimed for $(m_k)_{k\geq 1}$ by Lemma 3.2. This is now a well understood classical topic. See [M] for similar computations. Because of this rather than provide a detailed verification of this we content ourselves with the following brief remark.

Remark. Suppose R is a large positive integer to be chosen at our convenience. Let $M = \prod_{k=1}^{qR} m_k$ let $D = D(R, q) = p \max_{k \le R} \frac{m_{qk}}{m_k}$, where p is one of the primes dividing an element of $(m_k)_{k\ge1}^R$ and let P_l denote the semigroup of integers generated by the the first l primes. Here l is the smallest possible chosen so P_l contains all the products in the set $(m_k)_{k=1}^R$. Now let x be large and set

$$H = [x, Dx] \cap P_l \quad \text{with} \quad G = [m_q x, Dm_R] \cap MP_l\,,$$

where MP_l denotes the set of elements of P_l multiplied by M. Following [M] we observe that

$$\#H \sim K \log D\, (\log x)^{l-1} \quad \text{and} \quad \#G \sim \frac{Dm_R}{m_q}\Big(\log \frac{x}{M}\Big)^{s-1}.$$

These two observations readily imply condition (1) of Lemma 3.2. To demonstrate (2) of Lemma 3.2 set $G_\eta = \{g \in MP_l : \frac{g}{m_k} \in H \textit{ for all } r \in [\eta, q\eta]\}$ and note that $G_\eta = [m_{q\eta}x, m_\eta Dx] \cap MP_l$. We can check that $[m_{q\eta}x, m_\eta Dx]$ and $[m_{q(\eta+1)}x, m_{(\eta+1)}Dx]$ intersect. Thus $G = \cup_{\eta=1}^R G_\eta$ as required. We summarise this in the following lemma.

Lemma 3.3. *Suppose B is as just described by Lemma 3.2. If $(m_k)_{k\ge1}$ is not contained in a finitely generated subset of the natural numbers, then there is a G_δ set $B \subset S^1$ such that the averages*

$$\frac{1}{N}\sum_{k=1}^{N} f(\{m_k\theta\}) \qquad (N = 1, 2, \dots)$$

fail to converge almost everywhere in θ with respect to Lebesgue measure.

This proves part (b) of Theorem 2.4 and hence the whole theorem.

This final lemma has been known to the second author since 1991 hence his scepticism about the claim in [L]. It was also rediscovered independently by G. Kozma and this was communicated to the second author in 2002. A weaker L^1 counter example appears in [QW]. If $(m_k)_{k\ge1}$ is contained in a subset of the natural numbers generated multiplicatively by a finite set and $f \in L^1\big([0,1)\big)$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\{m_k\theta\}) = \int_0^1 f(t)dt \textit{ a.e. } \text{in } \theta,$$

with respect to Lebesgue measure is proved identically to Theorem 2.4 (a), using the idea in [N1] or [N2]. This observation also appears in [QW].

Acknowledgment. The second author thanks the Newton Institute in Cambridge for its hospitality while the final work on this paper was done. The authors also thank the referee for detailed comments which improved the presentation of the paper.

REFERENCES

[AMZ] AISTLEITNER, C.— MAYER, P.A.—ZIEGLER, V.: *Metric discrepancy theory, functions of bounded variation and gcd sums*, Uniform Distribution Theory **5** (2010), no. 1, 95–109.

[Ba] BAKER, R.C.: *Metric number theory and the large sieve*, J. London Math. Soc. (2) **24** (1981), no. 1, 34–40.

[Be] BEWLEY, T.: *Extension of the Birkhoff and von Neumann ergodic theorems to semigroup actions*, Ann. Inst. H. Poincar Sect. B (N.S.) **7** (1971), 283–291.

[DT] DRMOTA, M.—TICHY, R.F.: Sequences, Discrepancies and Applications. In: Lecture Notes in Math. **1951**, Springer Verlag, Berlin, 1997.

[dM] DE MATHAN, B.: *Numbers contravening a condition in density modulo 1* Acta Math. Acad. Sci. Hungar. **36** (1980), no. 3–4, 237–241

[KN] KUIPERS, L.—NIEDERREITER, H.: Uniform Distribution of Sequences, John Wiley & Sons, New York, 1974.

[L] LACROIX, Y.: *On strong uniform distribution. II*, Acta Arith. **84** (1998), no. 3, 279–290.

[M] MARSTRAND, J.M.: *On Khinchin's conjecture about strong uniform distribution*, Proc. London Math. Soc. (3) **21** (1970), 540–556.

[N1] NAIR, R.: Distributional Problems in Number Theory, PhD Thesis, University of Warwick 1986.

[N2] ______ , *Strong uniform distribution* Acta Arith. **56** (1990), no. 3, 183–193.

[N3] ______, *On affine maps and an arithmetic limit on compact groups*, Uniform Distribution Theory **4** (2009), no. 2, 39–45.

[P1] PHILIPP, W.: *Limit theorems for lacunary series and uniform distribution* mod1, Acta Arith. **26** (1974/75), no. 3, 241–251.

[P2] ______, *Empirical distribution functions and strong approximation theorems for dependent random variables. A problem of Baker in probabilistic number theory*, Trans. Amer. Math. Soc. **345** (1994), no. 2, 705–727.

[Po] POLLINGTON, A.D.: *On the density of sequence* $\{n_k\xi\}$, Illinois J. Math. **23** (1979), no. 4, 511–515.

[QW] QUAS, A.—WIERDL, M.: *Rates of divergence of non-conventional ergodic averages*, Ergodic Theory Dynam. Systems **30** (2010), no. 1, 233–262.

[R] RHIN, G.: *Sur la rpartition modulo 1 des suites f(p)*, Acta Arith. **23** (1973), 217–248.

[SZ] SCHMIDT, K.M.—ZHIGLJAVSKY, A.A.: *A characterization of the arcsine distribution*, Statist. Probab. Lett. **79** (2009), no. 24, 2451–2455.

[Wa] WALTERS, P.: *An Introduction to Ergodic Theory* Graduate texts in Math. **79** Springer-Verlag, 1982.

[We] WEYL, H.: *Über die Gleichverteilung von Zahlen mod. Eins* Math. Ann. **77**(1916), no. 3, 313–352.

Received February 23, 2013
Accepted July 14 , 2014

Kwo Chan
Radhakrishnan Nair
Mathematical Sciences
University of Liverpool
Liverpool L69 7ZL
UNITED KINGDOM
E-mail: sgkchan@liv.ac.uk
nair@liv.ac.uk