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ON THE DISTRIBUTION FUNCTIONS OF TWO OSCILLATING SEQUENCES

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ABSTRACT. We investigate the set of all distribution functions of two special sequences on the unit interval, which involve logarithmic and trigonometric terms. We completely characterize the set of all distribution functions $G(x_n)$ for $(x_n)_{n\geq 1} = (\{\cos(\alpha n)^n\})_{n\geq 1}$ and arbitrary α, where $\{x\}$ denotes the fractional part of x. Furthermore we give a sufficient number-theoretic condition on α for which $(x_n)_{n\geq 1} = (\{\log(n)\cos(\alpha n)\})_{n\geq 1}$ is uniformly distributed. Finally we calculate $G(x_n)$ in the case when $\frac{\alpha}{2\pi} \in \mathbb{Q}$.

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1. Introduction

In the present paper we consider the set of all distribution functions $G(x_n)$ of sequences $(x_n)_{n\geq 1}, x_n \in [0,1)$. For an interval $I \subseteq [0,1)$ we set $A(I,N,x_n)$ to be the number of hits of I among the first N elements of $(x_n)_{n\geq 1}$, i.e.,

$$A(I,N,x_n) = \#\{n \leq N : x_n \in I\} = \sum_{n=1}^{N} \mathbf{1}_I(x_n).$$

A non-decreasing function $g(x)$ satisfying $g(0) = 0, g(1) = 1$, is called a distribution function of a sequence $(x_n)_{n\geq 1}$ if there exists an increasing sequence $(N_k)_{k\geq 1}$ such that

$$g(x) = \lim_{k\to\infty} \frac{A([0,x), N_k, x_n)}{N_k}, \quad x \in [0,1], \tag{1}$$

holds in every point of continuity of g. In the sequel $G(x_n)$ denotes the set of all functions for which (1) holds.
Furthermore a sequence $(x_n)_{n\geq 1}$ is said to have the asymptotic distribution

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function $g(x)$ if (1) holds for $N_k = k$. Then the set $G(x_n)$ reduces to a singleton. Moreover if $G(x_n) = \{x\}$ the sequence $(x_n)_{n\geq 1}$ is called uniformly distributed (u.d.).

Closely connected to distribution functions of sequences is the concept of the discrepancy of sequences. For a sequence $(x_n)_{n\geq 1}$ the discrepancy of the first N elements of $(x_n)_{n\geq 1}$ is given as

$$D_N((x_n)_{n\geq 1}) = \sup_{0\leq\alpha<\beta\leq 1} \left| \frac{A([\alpha,\beta), N, x_n)}{N} - (\beta-\alpha) \right|.$$

Note that by a theorem of Weyl, see e.g. [11], it follows that

$$\lim_{N\to\infty} D_N\big((x_n)_{n\geq 1}\big) = 0$$

holds if and only if that the sequence $(x_n)_{n\geq 1}$ is u.d. in $[0,1)$. For general results on uniform distribution of sequences and discrepancy theory see [4], [8] or [11].

The following problem concerning the set of all distribution functions of a sequence is stated in the open problem section on the web site of *Uniform distribution theory* [1]:

> *Find the set $G(x_n)$ for the following sequences:*
>
> $$(x_n)_{n\geq 1} = (\{\cos(n)^n\})_{n\geq 1}, \tag{2}$$
> $$(x_n)_{n\geq 1} = (\{\log(n)\cos(\alpha n)\})_{n\geq 1}, \tag{3}$$
> $$(x_n)_{n\geq 1} = (\{\cos(n+\log(n))\})_{n\geq 1}, \tag{4}$$
>
> *where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.*

For the sequence in (4) this problem has already been solved by S. Steinerberger. One can find a short version of the proof in the open problem section of *Uniform distribution theory*[1]. The fact that the sequence in (4) is not u.d. has previously been proved by Kuipers, see [7].

The outline of the rest of the paper is as follows: in the second section we characterize the set of all distribution functions of a general version of the sequence given in (2). We use the Koksma inequality, see e.g. [11], in the third section to find a sufficient condition on the parameter α for which $(x_n)_{n\geq 1}$ given in (3) is u.d. Furthermore we give a complete solution of the problem in the case $\frac{\alpha}{2\pi} \in \mathbb{Q}$.

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[1] Problem 1.10 in the open problem collection as of 11. December 2011 (http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf)

2. The set of all distribution functions of $(\{\cos(\alpha n)^n\})_{n\geq 1}$

Former results on the sequence (2) are due to Bukor [3] and in more generality Luca [9]. Bukor showed that $(\cos(n)^n)_{n\geq 1}$ is everywhere dense in the interval $[-1,1]$. The following theorem characterizes the set of distribution functions $G(x_n)$ for

$$(x_n)_{n\geq 1} = (\{\cos(\alpha n)^n\})_{n\geq 1}\,,$$

which is a generalized version of (2).

Theorem 2.1. *For $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ we set $a = 3/4$, in the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ for p, q co-prime let*

$$a = \lim_{N\to\infty} \frac{\#\{n \leq N\colon (\cos(\alpha n))^n \geq 0\}}{N} = \begin{cases} \frac{q+1}{2q} + \frac{q-1}{4q}, & \text{if } 4 \mid (q-1),\\ \frac{q-1}{2q} + \frac{q+1}{4q}, & \text{if } 4 \nmid (q-1) \end{cases} \tag{5}$$

for q odd and let

$$a = \lim_{N\to\infty} \frac{\#\{n \leq N\colon (\cos(\alpha n))^n \geq 0\}}{N}$$

$$= \frac{\#\{1 \leq n \leq q\colon (\cos(\alpha n))^n \geq 0\}}{q} = \begin{cases} \frac{1}{2} + \frac{q-2}{4q}, & \text{if } 4 \nmid q \text{ and } 8 \mid (q-2),\\ \frac{1}{2} + \frac{q+2}{4q}, & \text{if } 4 \nmid q \text{ and } 8 \nmid (q-2),\\ \frac{q+2}{2q} + \frac{1}{4}, & \text{if } 4 \mid q \text{ and } 8 \nmid q,\\ \frac{q+2}{2q} + \frac{q-4}{4q}, & \text{if } 8 \mid q \end{cases} \tag{6}$$

for q even.
Then the set of all distribution functions of $(x_n)_{n\geq 1}$ is given by $G(x_n) = \{g_a(x)\}$, with

$$g_a(x) = \begin{cases} 0, & \text{if } x = 0,\\ a, & \text{if } 0 < x < 1,\\ 1, & \text{if } x = 1. \end{cases}$$

Proof. First we consider the case $\frac{\alpha}{2\pi} \notin \mathbb{Q}$. Let $\epsilon, \delta > 0$ be small and fixed. Then for sufficiently large $N_0 \in \mathbb{N}$ we have

$$\lambda(\{x \in [0,1]\colon \left|\cos(2\pi x)^{N_0}\right| > \delta\}) < \epsilon,$$

where λ denotes the Lebesgue measure. By the fact that $(\{\frac{\alpha}{2\pi} n\})_{n\geq 1}$ is u.d. in $[0,1)$ if $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ (see, e.g., [11]) it follows that

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : |\cos(\alpha n)^n| > \delta\}}{N} < \epsilon.$$

Furthermore since
$$\lambda(\{x \in [0,1]\colon \cos(2\pi x) < 0\}) = \frac{1}{2}$$
and by the fact that also $(\{(2\frac{\alpha}{2\pi}n\})_{n\geq 1}$ is u.d. in $[0,1)$ it follows that
$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n < 0\}}{N} = \frac{1}{4}.$$
Noting that $|\cos(\alpha n)^n| \leq \delta$ and $\cos(\alpha n)^n < 0$ imply $\{\cos(\alpha n)^n\} \geq 1-\delta$. Thus we have
$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n > \delta\}}{N} \geq \frac{3}{4} - \epsilon$$
and
$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n < 1-\delta\}}{N} \leq \frac{3}{4} + \epsilon$$
Since ϵ and δ are arbitrary, this proves the first part of the theorem.
In the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ we easily see that $(\{\cos(\alpha n)\})_{n\geq 1}$ takes only finitely many different values of the form $\cos\left(2\pi\frac{j}{q}\right), j = 1,\dots,q$, which appear periodically with period q.
Consider the case that q is odd. We want to calculate k given as
$$k = \#\left\{j \in \{1,\dots,q\}\colon \cos\left(2\pi\frac{j}{q}\right) \geq 0\right\}.$$
By $\cos\left(2\pi\frac{q}{q}\right) = 1$ and
$$\cos\left(2\pi\frac{j}{q}\right) = \cos\left(2\pi\frac{q-j}{q}\right), \qquad \text{for } j = 1,\dots,q,$$
it follows that $k = 2l-1$, where l is the maximal value in $\{1,\dots,q\}$ for which
$$2\pi\frac{l}{q} < \frac{\pi}{2} \Leftrightarrow l < \frac{q}{4}.$$
Thus if $4 \mid (q-1)$ we have $l = \frac{q-1}{4}$ and it follows $\cos\left(2\pi\frac{j}{q}\right) \geq 0$ for $\frac{q+1}{2}$ values of $j = 1,\dots,q$. Similarly if $4 \nmid (q-1)$ we have $l = \frac{q-3}{4}$ and $\cos\left(2\pi\frac{j}{q}\right) \geq 0$ for $\frac{q-1}{2}$ values of $j = 1,\dots,q$.
Since q is odd it follows for every $j = 1,\dots,q$ that the term $\cos\left(2\pi\frac{j}{q}\right)$ appears in $(\cos(\alpha n)^n)_{n\geq 1}$ alternating with odd and even exponent, thus for i such that $\cos(\alpha i) < 0$ it follows that
$$\lim_{N\to\infty} \frac{\#\{n \leq N\colon \cos(\alpha i)^{qn+i} > 0\}}{N} = \frac{1}{2}.$$
This, together with the above considerations, proves (5).

Assume now that q is even and $4 \nmid q$. Then $\cos(\alpha i) \neq 0$ for $i = 1, \dots, q$ and by similar considerations as above we have $\cos(\alpha i) > 0$ for $\frac{q}{2}$ values of i and $\frac{q}{2}$ is an odd number. Furthermore for every $j = 1, \dots, q$, the exponent of the term $\cos\left(2\pi\frac{j}{q}\right)$ in $(\cos(\alpha n)^n)_{n\geq 1}$ is either always odd or always even. Moreover it is easy to see that the exponents of $\cos\left(2\pi\frac{j}{q}\right)$ and $\cos\left(2\pi\frac{j+1}{q}\right)$ in $(\cos(\alpha n)^n)_{n\geq 1}$ cannot be both even or both odd. Thus the number of j's such that $\cos\left(2\pi\frac{j}{q}\right) < 0$ and which appear with even exponent is $\frac{q-2}{4}$ when $8 \mid (q-2)$ and $\frac{q+2}{4}$ when $8 \nmid (q-2)$.
In the case $4 \mid q$, it follows that $\cos(\alpha i) = 0$ for two values of i. Thus $\cos(\alpha i) \geq 0$ for $\frac{q+2}{2}$ values of $i = 1, \dots, q$. Formula (6) follows now by the similar arguments as above. This completes the proof. $\square$

3. The set of all distribution functions of $\left(\{\log(n)\cos(\alpha n)\}\right)_{n\geq 1}$

Properties of the sequence (3) have been studied by Berend, Boshernitzan and Kolesnik [1]. They proved the denseness of (3) in $[0, 1)$ for arbitrary α. Furthermore they also concluded in [2] that sequences of the form

$$\left(\{n^a \log^b(n) \cos(2\pi n\alpha)\}\right)_{n\geq 1} \tag{7}$$

are dense in $[0, 1)$, provided α is irrational and either $a > 0$ or $a = 0, b > 0$. Moreover they showed that the sequence in (7) is u.d. for α irrational and either $a > 0$ or $a = 0, b > 1$. Additionally they remarked without proof that the sequence (3) is not u.d. for uncountably many α, see [1].
The next theorem gives a condition on the parameter α which implies the u.d. property for the sequence in (3).

Theorem 3.1. *Let α be such that the discrepancy of the sequence $(z_n)_{n\geq 1} = \left(\{\frac{\alpha}{2\pi}n\}\right)_{n\geq 1}$ is of asymptotic order $o\left(\frac{1}{\log(N)}\right)$. Then the sequence*

$$(x_n)_{n\geq 1} = \left(\{\log(n)\cos(\alpha n)\}\right)_{n\geq 1}$$

is u.d.

Remark 3.1. It is well-known that there is a close connection between the coefficients of the continued fraction expansion of α and the asymptotic order of the discrepancy of $(\{\alpha n\})_{n\geq 1}$. By a classical result of Khintchine [6] from the metric theory of continued fractions, the discrepancy of $(\{\alpha n\})_{n\geq 1}$ satisfies

$$D_N\big(\{\alpha\},\dots,\{\alpha N\}\big) = \mathcal{O}\big(N^{-1}(\log N)(\log\log N)^{1+\epsilon}\big)$$

as $N\to\infty$ for almost all α. Thus the conclusion of Theorem 3.1 holds for almost all α.

Furthermore, if α is badly approximable, and in particular if α is an quadratic irrational such as $\alpha=\sqrt{2}$, then

$$D_N\big(\{\alpha\},\dots,\{\alpha N\}\big) = \mathcal{O}\big(N^{-1}\log N\big)$$

as $N\to\infty$, and thus the conclusion of Theorem 3.1 also holds for $\frac{\alpha}{2\pi}=\sqrt{2}$. For details see [8, p. 125, Th. 3.4].

For the proof of Theorem 3.1 we will use the following Lemma 3.1, which is a special case of the van der Corput lemma. It can be found e.g. in [8, Chapter 1, Section 1, Lemma 2.1].

Lemma 3.1. *Suppose that $\phi(x)$ is real-valued, that $|\phi'(x)|\geq\gamma$ for some positive γ, and that ϕ' is monotonic for all $x\in(\alpha,\beta)$. Then*

$$\left|\int_\alpha^\beta \cos\big(2\pi\phi(x)\big)\,dx\right|\leq\gamma^{-1}\qquad\text{and}\qquad\left|\int_\alpha^\beta \sin\big(2\pi\phi(x)\big)\,dx\right|\leq\gamma^{-1}.$$

Proof of Theorem 3.1:
To prove uniform distribution of $(x_n)_{n\geq 1}$, by the Weyl criterion (see [8]) it is sufficient to show that

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^N\cos(2\pi h x_n)\right|=0\qquad\text{and}\qquad\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^N\sin(2\pi h x_n)\right|=0 \tag{8}$$

holds for all $h>0$. We will only prove (8) for cosine-functions, the case of sine-functions being entirely similar. Assume that $h>0$ is fixed. Choose $\epsilon>0$ and $\delta>0$ "small", and set

$$K=\frac{-\log(\epsilon)}{\log(1+\delta)}.$$

For simplicity of writing, we assume that K is an integer. For $N\geq 1$, we define

$$m_i=\left\lceil \epsilon N(1+\delta)^i\right\rceil,\qquad 0\leq i\leq K,$$

and

$$y_n=\log(\epsilon N(1+\delta)^i)\cos(\alpha n)\qquad\text{for}\qquad m_{i-1}<n\leq m_i,\quad 1\leq i\leq K.$$

Note that $m_0 = \lceil \epsilon N \rceil$ and $m_K = \epsilon N(1+\delta)^K = N$. We have

$$\left| \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi h x_n) \right| \tag{9}$$

$$\leq \left| \frac{1}{N} \sum_{n=1}^{m_0} \cos(2\pi h x_n) \right| + \left| \frac{1}{N} \sum_{n=m_0+1}^{N} \big(\cos(2\pi h x_n) - \cos(2\pi h y_n) \big) \right| \tag{10}$$

$$+ \left| \frac{1}{N} \sum_{n=m_0+1}^{N} \cos(2\pi h y_n) \right|. \tag{11}$$

The first term in line (10) is trivially bounded by $m_0/N = \lceil \epsilon N \rceil / N$. Now we turn to the second term in (10). For any $i \in \{1, \dots, K\}$ and $m_{i-1} < n \leq m_i$ we have

$$\begin{aligned}
|\cos(2\pi h x_n) - \cos(2\pi h y_n)| &= |\cos(2\pi h u_n) - \cos(2\pi h y_n)| \\
&\leq 2\pi h \, |u_n - y_n| \\
&= 2\pi h \left| \log(n) \cos(\alpha n) - \log\big(\epsilon N (1+\delta)^i\big) \cos(\alpha n) \right| \\
&\leq 2\pi h \left(\lceil \log(\epsilon N (1+\delta)^i \rceil - \log(n) \right) |\cos(\alpha n)| \\
&\leq 2\pi h \left(\lceil \log(\epsilon N (1+\delta)^i \rceil - \lceil \log(\epsilon N (1+\delta)^{i-1} \rceil - 1 \right) \\
&\leq 2\pi h \log(1+\delta).
\end{aligned}$$

where $u_n = \log(n) \cos(\alpha n)$. Thus we get

$$\left| \frac{1}{N} \left(\sum_{n=\epsilon N+1}^{N} (\cos(2\pi h x_n) - \cos(2\pi h y_n)) \right) \right| \leq (2\pi h)^2 \log(1+\delta). \tag{12}$$

Finally we estimate the term in line (11). For $1 \leq i \leq K$ we set

$$f_i(x) = \cos \Big(2\pi h \big(\log(\epsilon N (1+\delta)^i) \cos(2\pi x) \big) \Big).$$

The total variation $\mathrm{Var}(f_i)$ of this function on the interval $[0,1]$ is at most

$$8h \lceil \log(\epsilon N (1+\delta)^i) \rceil. \tag{13}$$

For any number $R > 1$, in the interval $[R^{-1/2}, 1/4 - R^{-1/2}]$ the derivative of the function $R\cos(2\pi x)$ is monotonous, and of absolute value at least $R^{1/2}/2$. Thus, using Lemma 3.1 for the function $f_i(x) = \cos(2\pi \Phi(x))$, where $\Phi(x) = R \cos(2\pi x)$

and the previous remark for $R = h\big(\log(\epsilon N(1+\delta)^i)$, we have

$$\begin{aligned}
\left|\int_0^{1/4} f_i(x)\,dx\right| &\leq \underbrace{\int_0^{R^{-1/2}} |f_i(x)|\,dx + \int_{1/4-R^{-1/2}}^{1/4} |f_i(x)|\,dx}_{\leq 2R^{-1/2}} \\
&+ \underbrace{\left|\int_{R^{-1/2}}^{1/4-R^{-1/2}} f_i(x)\,dx\right|}_{\leq 2R^{-1/2}} \\
&\leq 4h^{-1/2}\Big(\log\big(\epsilon N(1+\delta)^i\big)\Big)^{-1/2}.
\end{aligned}$$

A similar estimate holds for the absolute value of $\int_{i/4}^{(i+1)/4} f_i(x)\,dx,\ i = 1,2,3,$ and in total we obtain

$$\left|\int_0^1 f_i(x)\,dx\right| \leq 16h^{-1/2}\Big(\log\big(\epsilon N(1+\delta)^i\big)\Big)^{-1/2} \leq 16\big(\log(\epsilon N)\big)^{-1/2}, \quad 1 \leq i \leq K. \tag{14}$$

For the term in line (11) we have

$$\left|\frac{1}{N}\left(\sum_{n=\lceil\epsilon N\rceil+1}^{N} \cos(2\pi h y_n)\right)\right| \leq \sum_{i=1}^{K} \frac{\Delta m_i}{N}\left|\frac{1}{\Delta m_i}\left(\sum_{n=m_{i-1}+1}^{m_i} \cos(2\pi h y_n)\right)\right|, \tag{15}$$

where $\Delta m_i = m_i - m_{i-1}$. Note that we can write

$$\cos(\alpha n) = \cos\left(2\pi\left(\left\lfloor\frac{\alpha n}{2\pi}\right\rfloor + \left\{\frac{\alpha n}{2\pi}\right\}\right)\right) = \cos\left(2\pi\left\{\frac{\alpha n}{2\pi}\right\}\right) = \cos(2\pi z_n).$$

Thus for any $i,\ 1 \leq i \leq K$, using Koksma's inequality for the function $f_i(x)$, together with (13) and (14), we get

$$\begin{aligned}
&\left|\frac{1}{m_i - m_{i-1}}\left(\sum_{n=m_{i-1}+1}^{m_i} \cos(2\pi h y_n)\right)\right| \\
= &\left|\frac{1}{m_i - m_{i-1}}\left(\sum_{n=m_{i-1}+1}^{m_i} \cos\Big(2\pi h \log\big(\epsilon N(1+\delta)^i\big)\cos(2\pi z_n)\Big)\right)\right| \\
\leq &\left|\int_0^1 f_i(x)\,dx\right| + \mathrm{Var}(f_i) D_{m_i - m_{i-1}}(z_{m_{i-1}+1}, \ldots, z_{m_i}) \\
\leq &\ 16(\log(\epsilon N))^{-1/2} + 8h\lceil\log(\epsilon N(1+\delta)^i)\rceil D_{m_i - m_{i-1}}(z_{m_{i-1}+1}, \ldots, z_{m_i}). \qquad (16)
\end{aligned}$$

By the triangle inequality for discrepancies and the assumption on the discrepancy of $(z6_n)_{n\geq 1}$,

$$\begin{aligned} & D_{m_i - m_{i-1}}(z_{m_{i-1}+1}, \ldots, z_{m_i}) \\ \leq \; & \frac{m_{i-1}}{m_i - m_{i-1}} D_{m_{i-1}}(z_1, \ldots, z_{m_{i-1}}) + \frac{m_i}{m_i - m_{i-1}} D_{m_i}(z_1, \ldots, z_{m_i}) \\ \leq \; & \frac{\epsilon}{\log N} \end{aligned} \tag{17}$$

for all $i \in \{1, \ldots, K\}$, provided N is sufficiently large. Using (16) and (17), we thus see that (15) is at most

$$16\big(\log(\epsilon N)\big)^{-1/2} + 8h\epsilon \frac{\lceil \log N \rceil}{\log N} \tag{18}$$

for sufficiently large N.

Combining all our estimates for (9), we finally obtain

$$\left| \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi h x_n) \right| \leq \frac{\lceil \epsilon N \rceil}{N} + (2\pi h)^2 \log(1+\delta) + 16\big(\log(\epsilon N)\big)^{-1/2} + 8h\epsilon \frac{\lceil \log N \rceil}{\log N}$$

for suffiently large N. Since ϵ and δ can be chosen arbitrarily small, this proves the theorem. □

The following lemma by Pólya and Szegö [10] characterizes the set of distribution functions $G(\{c \log(n)\})$ and is the main tool for the proof of Theorem 3.2.

Lemma 3.2 (Pólya and Szegö)**.** *The sequence $(x_n)_{n\geq 1} = (\{c \log n\})_{n \geq 1},\ c > 0$, has distribution functions of the form*

$$g_{\beta,c}(x) = \frac{e^{\frac{\min(x,\beta)}{c}} - 1}{e^{\frac{\beta}{c}}} + \frac{1}{e^{\frac{\beta}{c}}} \frac{e^{\frac{x}{c}} - 1}{e^{\frac{1}{c}} - 1}, \tag{19}$$

where $\lim_{k\to\infty} \{c \log N_k\} = \beta$ implies $F_{N_k}(x) \to g_\beta(x)$ and $F_N(x) = \frac{\#\{n \leq N; x_n \in [0,x)\}}{N}$. Moreover $G(\{c \log(n)\})$ is the set of all distribution functions of the form (19).

Remark 3.2. Note that it follows as a corollary of Lemma 3.2 that the sequence $(x_n)_{n\geq 1} = (\{c \log(n)\})_{n\geq 1}$, $c < 0$, has distribution functions of the form $1 - g_{\beta,|c|}(1-x)$, where $\lim_{k\to\infty}\{|c| \log N_k\} = \beta$ and $g_{\beta,|c|}(x)$ is given in (19).

Thus we define the function $f_{\beta,c}(x)$ as

$$f_{\beta,c}(x)=\begin{cases} g_{\beta,c}(x), & \text{if } c>0,\\ 1-g_{\beta,|c|}(1-x) & \text{if } c<0,\\ \mathbf{1}_{\{(0,1]\}}(x) & \text{if } c=0,\end{cases} \tag{20}$$

where $\lim_{k\to\infty}\{|c|\log N_k\}=\beta$.

Theorem 3.2. *Let $\frac{p}{q}:=\frac{\alpha}{2\pi}\in\mathbb{Q}$ where p,q are co-prime and*

$$(x_n)_{n\ge 1}=\big(\{\log(n)\cos(\alpha n)\}\big)_{n\ge 1}.$$

Then for a fixed subsequence $(N_k)_{k\ge 1}$ of $\mathbb{N}$ with

$$\lim_{k\to\infty}\{\cos(\alpha i)\log N_k\}=\beta_i,\qquad \textit{for } i=1,\dots,q, \tag{21}$$

the asymptotic distribution of $(x_n)_{n\ge 1}$ along the subsequence $(N_k)_{k\ge 1}$ is given by

$$f(x)=\frac{1}{q}\sum_{i=1}^{q}h_{q,\beta_i,c_i}(x), \tag{22}$$

where $h_{q,\beta_i,c_i}(x)$ is given in (20),

$$h_{q,\beta_i,c_i}(x)= \tag{23}$$
$$\begin{cases} f_{\beta_i,c_i}(x+1-\nu_i)-f_{\beta_i,c_i}(1-\nu_i), & \textit{if } 0\le x\le\nu_i \textit{ and } c_i>0,\\ f_{\beta_i,c_i}(x-\nu_i)+1-f_{\beta_i,c_i}(1-\nu_i), & \textit{if } \nu_i\le x\le 1 \textit{ and } c_i>0,\\ f_{\beta_i,c_i}(x+\nu_i)-f_{\beta_i,c_i}(\nu_i), & \textit{if } 0\le x\le 1-\nu_i \textit{ and } c_i<0,\\ f_{\beta_i,c_i}(x-(1-\nu_i))+1-f_{\beta_i,c_i}(\nu_i), & \textit{if } 1-\nu_i\le x\le 1 \textit{ and } c_i<0,\\ \mathbf{1}_{\{(0,1]\}}(x) & \textit{if } c_i=0,\end{cases}$$

where $\nu_i=\{|c_i|\log(q)\}$, $c_i=\cos(\alpha i)$, $\lim_{n\to\infty}\{\cos(\alpha i)\log N_k\}=\beta_i$ and $f_{\beta,c}(x)$ is given in (20).
Moreover, the set $G(x_n)$ is the set of all distribution functions of the form (22) for those $(\beta_1,\dots,\beta_q)$ for which a subsequence $(N_k)_{k\ge 1}$ satisfying (21) exists.

Remark 3.3. For arbitrary q, it is a difficult problem to determine all possible vectors $(\beta_1,\dots,\beta_q)$ for which there exists a subsequence $(N_k)_{k\ge 1}$ for which (21) holds, because there can exist non-trivial linear relations between the values $\cos(\alpha i), i=1,\dots,q$ (depending on number-theoretic properties of q). We will not further investigate this issue, the interested reader is referred to a Galois theoretic approach to this problem by Girstmair [5].

Proof. As mentioned in the proof of Theorem 2.1 the function $\cos(\alpha n), n\in\mathbb{N}$ takes only finitely many different values which appear in a period of length q.

Let $(N_k)_{k\geq 1}$ be a sequence of $\mathbb{N}$ for which (21) holds for some numbers $\beta_1,\dots,\beta_q$. We are interested in the asymptotic behavior of $(x_n)_{n\geq 1}$ and by

$$\begin{aligned}\lim_{k\to\infty}\frac{A([a,b),N_k,x_n)}{N_k} &= \lim_{k\to\infty}\frac{\sum_{i=1}^{q} A([a,b),\lfloor N_k/q\rfloor,\cos(\alpha i)\log(qn+i))}{N_k}\\ &= \lim_{k\to\infty}\frac{\sum_{i=1}^{q} A([a,b),\lfloor N_k/q\rfloor,\cos(\alpha i)\log(qn))}{N_k},\end{aligned}$$

we get that the limit distribution of $(x_n)_{n\geq 1}$ is a mixture of limit distributions of q sequences of the form

$$(z_n^i)_{n\geq 1} = \big(\{\log(qn)\cos(\alpha i)\}\big)_{n\geq 1}, \qquad \text{for } i\in\{1,\dots,q\}.$$

Since $\log(qn)\cos(\alpha i) = \log(q)\cos(\alpha i) + \log(n)\cos(\alpha i)$ we get that the distribution function of $(z_n^i)_{n\geq 1}$ is given as the distribution function of $\log(n)\cos(\alpha i)$ shifted by a constant and thus it is easy to see that the right hand-side of (23) is the distribution function of $(z_n^i)_{n\geq 1}$. This proves (22).
In order to prove that all functions in $G(x_n)$ can be characterized by (22) we use the Bolzano-Weierstrass theorem. Assume that $(\{\cos(\alpha i)\log N_k\})_{k\geq 1}$ does not converge for at least one i, then it follows that there are at least two convergent subsequences of $(\{\cos(\alpha i)\log N_k\})_{k\geq 1}$ which have different limits. If the limit distributions along all these subsequences are equal then the limit distribution of $(\{\cos(\alpha i)\log N_k\})_{k\geq 1}$ along the whole sequence $(N_k)_{k\geq 1}$ is (if it exists) can be written as $f_{\beta,c}(h(x))$ as the limit along a sequence for which (21) holds. If at least two limit distributions along subsequences are not equal the limit distribution along $(\{\cos(\alpha i)\log N_k\})_{k\geq 1}$ does not exist. □

Remark 3.4. We illustrate the set $G(x_n)$ in the simple case when $\alpha=\pi$, which means that $p=1$ and $q=2$ in the notation of Theorem 3.2. Note that $\cos(\alpha)=-1$ and $\cos(2\alpha)=1$, thus as a sequence $(N_k)_{k\geq 1}$ which satisfies (21) one can choose for example $N_k=\lfloor\exp(k+\beta_1)\rfloor$. In this case we have

$$\lim_{k\to\infty}\{\cos(\alpha)\log(N_k)\} = \beta_1$$

and

$$\lim_{k\to\infty}\{\cos(2\alpha)\log(N_k)\} = \beta_2 = 1-\beta_1.$$

We can calculate the corresponding distribution function by using the previous theorem. Figure 1 illustrates the range of distribution functions which one can achieve by varying β.

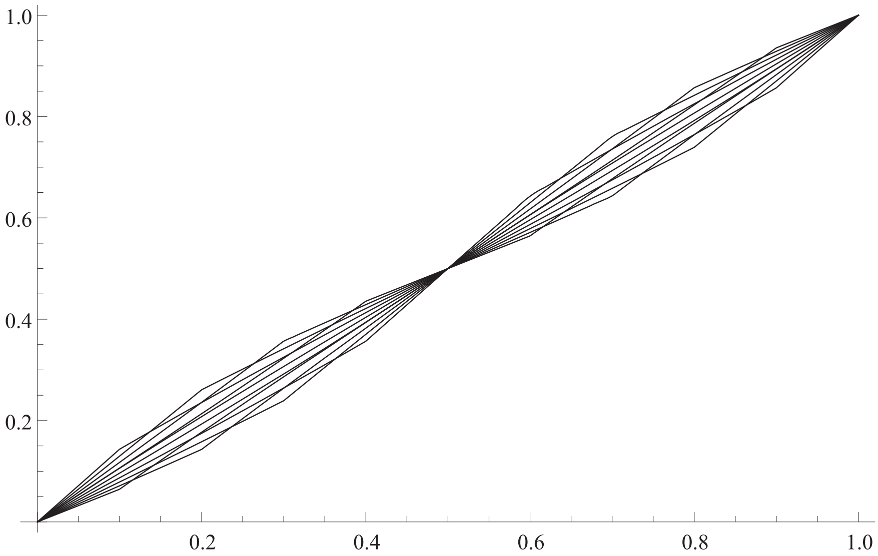

FIGURE 1. Distribution functions of $(x_n)_{n\geq 1}$ for $\alpha = \pi$ and $\beta = 0, \frac{1}{10}, \dots, \frac{9}{10}$.

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