

SYMMETRIES IN RAUZY FRACTALS

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ABSTRACT. In the present paper we study geometrical symmetries of the Rauzy fractals and their relation to symbolic symmetries, i.e. symmetries of the languages that define the fractals. The geometrical symmetries studied here are reflections through a point, i.e. its center of symmetry. We show that for unimodular Pisot substitutions so that the abelianization of the set of proper prefixes is symmetric in $\mathbb{R}$, there is a symmetrical subset of the Rauzy fractal. This subset corresponds to the maximal symbolic system in the paths of the prefix automaton that is invariant under the involution defined by the symmetries of the prefixes. We also give a generalization of this construction when the abelianization of the set of proper prefixes is not symmetrical.

We apply some of these techniques to show that the Rauzy fractal of substitutions of type

$$1 \rightarrow \underbrace{1 \cdots 1}_n 2, \quad 2 \rightarrow \underbrace{1 \cdots 1}_n 3, \quad \dots, \quad (k-1) \rightarrow \underbrace{1 \cdots 1}_n k, \quad k \rightarrow 1;$$

with $n \geq 1$ and $k \geq 3$, are symmetric.

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This paper is dedicated to the memory of G. Rauzy.

1. Introduction

The Rauzy fractal is an important object in the study of the dynamical systems associated to the Pisot substitutions, in particular the Pisot conjecture. Geometrical properties of the Rauzy fractals have been studying extensively, see among other references [1, 3, 5, 10, 12, 14, 16, 21, 23]. In the present paper we study the symmetries in Rauzy fractals and its relation to symbolic symmetries, i.e. symmetries in the language which defines the fractal. In [20] the

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author explored the relation between symbolic and geometrical symmetries for the k -bonacci substitutions.

The Rauzy fractal is defined by a language obtained by an automaton, i.e. the prefix automaton of the substitution. We shall show how the symmetries of this language are related to the geometrical symmetries of the Rauzy fractal. In the language generated by the prefix automaton of the substitution, we consider the maximal symmetric sub-language defined by an involution. This symmetric language is generated by another automaton, the so called canonical product. This symmetric language allows us to obtained a symmetric subset of the Rauzy fractal. More precisely, for any unimodular Pisot substitution so that the abelianization of the set of proper prefixes, i.e. the embedding of the set of proper prefixes in $\mathbb{R}$ (see Section 2 for precise definitions), is symmetric, we construct a symmetric subset of the Rauzy fractal, Theorem 3.1. This set is the “geometrical realization” of the maximal symbolic system in the paths of the prefix automaton such that is invariant under the involution defined by the symmetry of the prefixes. In Section 4 we give a generalization of this construction when the abelianization of the set of proper prefixes is not symmetric.

Later we consider the family of substitutions

$$\zeta = \zeta_{k,n} : \left\{ \begin{array}{ll} 1 & \rightarrow \underbrace{1 \cdots 1}_n 2 \\ 2 & \rightarrow \underbrace{1 \cdots 1}_n 3 \\ & \vdots \\ k-1 & \rightarrow \underbrace{1 \cdots 1}_n k \\ k & \rightarrow 1 \end{array} \right. \quad (1)$$

with $n \geq 1$ and $k \geq 3$. If $n = 1$ the corresponding substitution is called k -bonacci, whose symmetries are studied in [20]. In [23] some geometrical properties of the Rauzy fractals of these substitutions, with $k = 3$, were studied. We show in Theorem 3.2, that the Rauzy fractals for these substitutions are symmetric and they contain their centers of symmetry, Corollary 3.3. In Section 4, we list some basic properties of the geometry of the “geometrical realization” of canonical product of these substitutions. In Section 4 we point out that there are substitutions so that the embedding of the set of proper prefixes is symmetric, however the Rauzy fractal is not symmetric.

2. Substitutions and Automata

A substitution on a finite alphabet $\mathcal{A} = \{1, \dots, k\}$ is a map ζ from $\mathcal{A}$ to the set of finite words on $\mathcal{A}$, i.e. $\mathcal{A}^* = \cup_{i \geq 0} \mathcal{A}^i$. The map ζ is extended to $\mathcal{A}^*$ by concatenation, i.e. $\zeta(\emptyset) = \emptyset$ and $\zeta(UV) = \zeta(U)\zeta(V)$, for all $U, V \in \mathcal{A}^*$. Let $\mathcal{A}^{\mathbb{N}}$ (respectively $\mathcal{A}^{\mathbb{Z}}$) denote the set of one-sided (respectively two-sided) infinite sequences in $\mathcal{A}$. The map ζ , is extended to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ in the obvious way. We call $u \in \mathcal{A}^{\mathbb{N}}$ (or $u \in \mathcal{A}^{\mathbb{Z}}$) a *fixed point* of ζ if $\zeta(u) = u$ and *periodic* if there exists $l > 0$ so that it is fixed for ζ^l .

The *incidence matrix* of the substitution ζ is defined as the matrix $M = (m_{ij})$ whose entries m_{ij} is the number of occurrences of the symbol i in the word $\zeta(j)$, for $1 \leq i, j \leq k$. We say the substitution is *primitive* if its incidence matrix is primitive, i.e. all the entries of M^l are positive for some $l > 0$.

Let λ be the Perron-Frobenius eigenvalue of the incidence matrix M and $\ell = (\ell_i)_{1 \leq i \leq k}$ the associated positive left eigenvector, which we normalized by setting $\ell_1 = 1$.

For a primitive substitution there are a finite number of periodic points. So without loss of generality we can assume that a primitive substitution has always a fixed point. If ζ has only periodic points we consider ζ^l , for l so that it has a fixed point.

Let u be a fixed point of ζ , we consider the dynamical system (Ω_u, σ) , where σ is the shift map on $\mathcal{A}^{\mathbb{N}}$ (respectively on $\mathcal{A}^{\mathbb{Z}}$) defined by $\sigma(v_0 v_1 \dots) = v_1 \dots$ (respectively $\sigma(v) = w$, where $w_i = v_{i+1}$) and Ω_u is the closure of the orbit of the fixed point u under the shift map σ .

A substitution is *Pisot* if the Perron-Frobenius eigenvalue of the the incidence matrix is a Pisot number and the characteristic polynomial is irreducible. i.e. all non-Perron-Frobenius eigenvalues are nonzero and less than one in modulus. A Pisot substitution is primitive [5]. A substitution is *unimodular Pisot* if it is Pisot and the absolute value of the determinant of its incidence matrix is 1. Throughout this article we will consider only unimodular Pisot substitutions.

There is a long standing conjecture that the dynamical system associated to a unimodular Pisot substitution is measurably conjugate to a translation on a $(k-1)$ -dimensional torus (cf. [14, 22, 24]). This conjecture is known in literature as the Pisot conjecture. G. Rauzy approached it via geometrical realization of the symbolic system. He proved it in the case of the tribonacci substitution, $\zeta(1) = 12$, $\zeta(2) = 13$ and $\zeta(3) = 1$ (cf. [14]). In his proof, the construction of a set in $\mathbb{R}^2$, in general $\mathbb{R}^{k-1}$, plays an important rôle. This set is known as the Rauzy fractal associated to the substitution.

An alternative way to represent a substitution is by an automaton. Let $W \subset \mathcal{A}^* = \cup_{i \geq 0} \mathcal{A}^i$ be a set of finite words on the alphabet $\mathcal{A} = \{1, \dots, k\}$. An *automaton* over $\mathcal{A}$, $\mathfrak{A} = (Q, W, E, I)$ is a direct graph labelled by elements of $\mathcal{A}$. Q is the set of *states*, $I \subset Q$ is the set of *initial states*, W is the set of *labels* and $E \subset Q \times W \times Q$ is the set of labelled *edges* or *transitions*. If $(p, w, q) \in E$ we say that w is a transition between p and q .

The prefix automaton of the substitution ζ (cf. [15]) is the automaton $\mathfrak{A} = (Q, W, E, I)$ so that

- (i) $Q = \mathcal{A}$;
- (ii) $W = \text{Pref}$, the set of proper prefixes of the words $\zeta(i)$, for $i \in \mathcal{A}$. We shall denote by $\emptyset$ the empty prefix;
- (iii) (p, w, q) is in E if p, q are elements of $\mathcal{A}$, $w \in W$ and wq is a prefix of $\zeta(p)$.
- (iv) $I = \{1\}$.

The automaton reads words from left to right.

We change this automaton slightly: Define the map $l : \mathcal{A}^* \rightarrow \mathbb{Q}(\lambda)$ by $l(w_1 \dots w_m) := \ell_{w_1} + \dots + \ell_{w_m}$, with $l(\emptyset) = 0$. Then we denote the set $\{l(w) : w \in \text{Pref}\}$ by $l(\text{Pref})$ or W' . We use $l(\text{Pref})$ as set of labels and transitions are labelled by $l(w)$. We call this automaton the *modified prefix automaton of the substitution* ζ .

We remark that the ℓ_i are rational independent for a Pisot substitution. We embedded $\mathbb{Q}(\lambda)$ in $\mathbb{R}$ in the natural way.

The modified prefix automaton of the substitution of type (1) is in Figure 1. In this figure multiple transitions between the same states are represented on the same edge and separated by commas.

A *finite path in the automaton*, $\mathfrak{A}$, is word in E , the set of transitions: $(p_m, a_m, q_m)(p_{m-1}, a_{m-1}, q_{m-1}) \dots (p_0, a_0, q_0)$ such that $q_i = p_{i-1}$ for $1 \leq i \leq m$ and $p_m \in I$; however we usually denote the paths using only the labels, i.e. $a_m \dots a_0$. See [7] for more on automata theory.

Let

$$\mathcal{R} := \{\mathbf{a} = a_0 a_1 \dots \in W^{\mathbb{N}} : a_m \dots a_0 \text{ is a path in } \mathfrak{A} \text{ for all } m \in \mathbb{N}\},$$

where $\mathfrak{A}$ is the modified prefix automaton described before. We consider in $\mathcal{R}$ the topology induced from the product topology of $W^{\mathbb{N}}$.

We say that a set $\Omega \subset \mathbb{R}^d$ is *symmetric as subset of* $\mathbb{R}^d$, if there exists $L \in \mathbb{R}^d$ such that for any $\omega \in \Omega$, there exists $\hat{\omega} \in \Omega$ with $\omega + \hat{\omega} = L$, i.e. it is invariant under the reflection through the point $L/2$. This point, $L/2$ is called the *center of symmetry* of Ω .

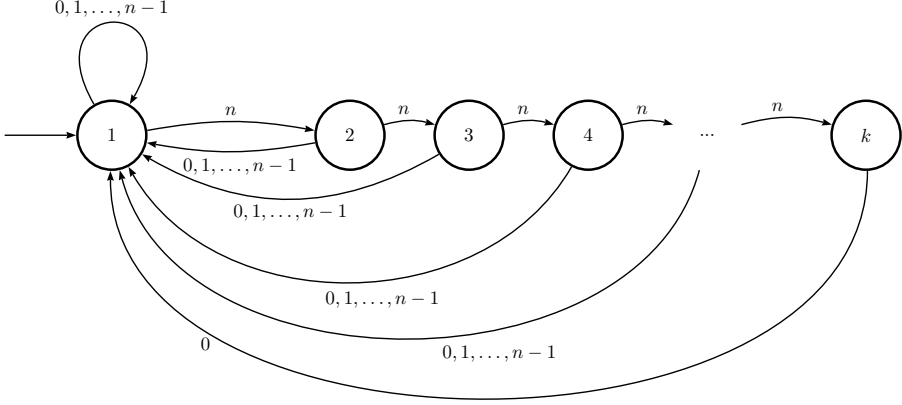


FIGURE 1. The automaton of the substitution of type (1).

If $l(\text{Pref})$ is symmetric as a subset of $\mathbb{R}$ we define the *dual automaton* of

$$\mathfrak{A} = (Q, W', E, I), \quad \text{as} \quad \widehat{\mathfrak{A}} = (Q, W', \widehat{E}, I)$$

where $\widehat{E} = \{(p, \widehat{w}, q) : (p, w, q) \in E\}$ where $\widehat{w} = L - w$. Similarly, we can consider the set

$$\widehat{\mathcal{R}} := \left\{ \mathbf{a} = a_0 a_1 \dots \in W'^{\mathbb{N}} : a_m \dots a_0 \text{ is a path in } \widehat{\mathfrak{A}} \text{ for all } m \in \mathbb{N} \right\}.$$

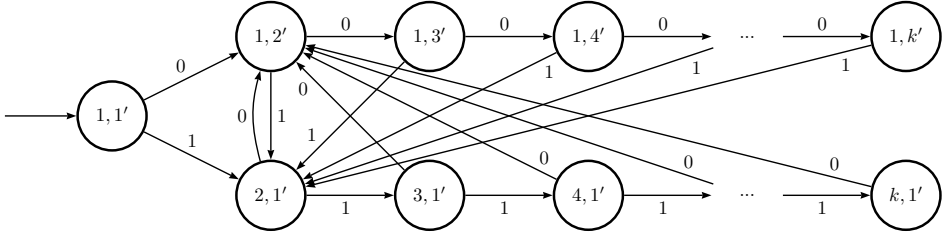
2.1. Product Automaton

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two automata on the alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. The *product automaton* $\mathfrak{A}_1 \times \mathfrak{A}_2$ is defined as the automaton as:

- set of states: $\mathcal{A}_1 \times \mathcal{A}_2$.
- labels: w is a label of $\mathfrak{A}_1 \times \mathfrak{A}_2$ if and only if w is a label of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.
- transitions: $((p_1, p_2), w, (q_1, q_2))$ is a transition in $\mathfrak{A}_1 \times \mathfrak{A}_2$ if and only if (p_i, w, q_i) is a transition in $\mathfrak{A}_i$, for $i = 1$ and 2 .
- Initial states: The product of the initial states of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

And finally the connected component containing the initial state is taken.

The paths in $\mathfrak{A}_1 \times \mathfrak{A}_2$ are paths in $\mathfrak{A}_1$ and $\mathfrak{A}_2$, so it is said that the paths in the product automaton are the common paths in each of the factors. The product automaton was used by the author, in [19, 17], in the context of substitution dynamical systems.


 FIGURE 2. The canonical product of k -bonacci automaton, i.e. $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$.

Let $\mathfrak{A}$ be the modified prefix automaton of a substitution of so that $l(\text{Pref})$ is symmetric as a subset of $\mathbb{R}$. The product automaton $\mathfrak{A} \times \widehat{\mathfrak{A}}$ is called the *canonical product* of $\mathfrak{A}$. In Section 4 we give a definition of the canonical product when $l(\text{Pref})$ is not symmetric.

Let

$$\mathcal{P} := \left\{ \mathbf{a} = a_0 a_1 \dots \in W^{\mathbb{N}} : a_m \dots a_0 \text{ is a path in } \mathfrak{A} \times \widehat{\mathfrak{A}} \text{ for all } m \in \mathbb{N} \right\}.$$

Let $\Psi : W^{\mathbb{N}} \rightarrow W^{\mathbb{N}}$ be the involution $\Psi(\mathbf{a}) = \widehat{\mathbf{a}} = \widehat{a_0} \widehat{a_1} \widehat{a_2} \dots$. The set $\mathcal{P}$ is the maximal subset of $\mathcal{R}$ invariant under Ψ , in the sense that any Ψ -invariant subset of $\mathcal{R}$ is a subset of $\mathcal{P}$. In fact: Let $\mathcal{B}$ be a Ψ -invariant subset of $\mathcal{R}$ and $\mathbf{a} = a_0 a_1 \dots$ one of its elements. So $a_m \dots a_0$ and $\widehat{a_m} \dots \widehat{a_0}$ are paths in the automata $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$, respectively, for all $m \in \mathbb{N}$. Since $\mathcal{B}$ is Ψ -invariant, $\widehat{a_m} \dots \widehat{a_0}$ is path in $\mathfrak{A}$, for all m . Using that Ψ is an involution, we get that $a_m \dots a_0$ is a path in $\widehat{\mathfrak{A}}$, for all m . Therefore $a_m \dots a_0$ is a path in the automaton $\mathfrak{A} \times \widehat{\mathfrak{A}}$ for all $m \in \mathbb{N}$. Hence $\Psi(\mathbf{a}) \in \mathcal{P}$.

3. Rauzy Fractals

Let ζ be an unimodular Pisot substitution and M its incidence matrix. We order the eigenvalues of M as follows:

$$\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1}, \overline{\lambda_{r+1}}, \lambda_{r+2}, \overline{\lambda_{r+2}}, \dots, \lambda_{r+s}, \overline{\lambda_{r+s}}$$

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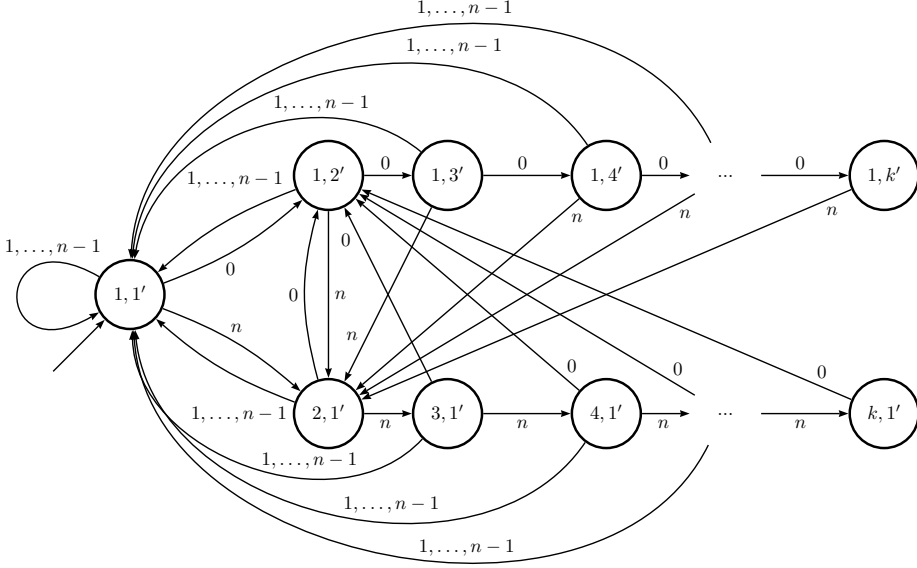


FIGURE 3. The canonical product of the substitution of type (1), with $n > 1$.

where $\lambda = \lambda_1$ is the Perron-Frobenius eigenvalue of M , the first r are distinct real eigenvalues and the last $2s$ are pairs of complex conjugate eigenvalues,

$$r + 2s = k.$$

We identify

$$\mathbb{R}^{r-1} \times \mathbb{C}^s \quad \text{with} \quad \mathbb{R}^{k-1}.$$

The collection of the Galois automorphisms

$$\lambda_1 \mapsto \lambda_i$$

defines the star map by

$$\begin{aligned} .* : \mathbb{Q}(\lambda_1) &\rightarrow \mathbb{R}^{r-1} \times \mathbb{C}^s, \\ \lambda_1 &\mapsto (\lambda_2, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_{r+s}). \end{aligned}$$

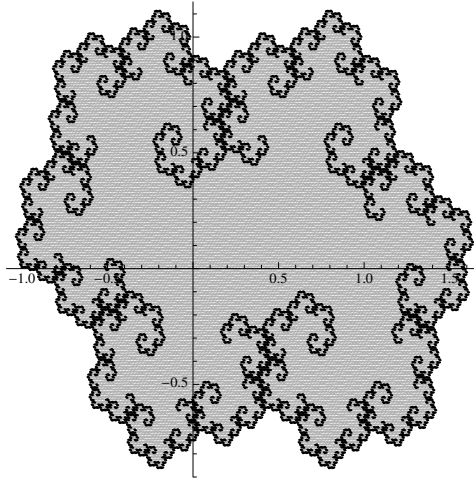


FIGURE 4. The Rauzy fractal of the substitution $\zeta_{3,1}$ and its symmetrical subset $\mathfrak{F}$ in black.

Let $\xi : W^{\mathbb{N}} \rightarrow \mathbb{R}^{r-1} \times \mathbb{C}^s$ be defined by $\xi(\mathbf{a}) = \sum_{i \geq 0} (a_i \lambda^i)^*$. The *Rauzy fractal* associated to the substitution ζ is the set $\mathfrak{R} := \xi(\mathcal{R})$, i.e.

$$\mathfrak{R} = \left\{ \sum_{i \geq 0} (a_i \lambda^i)^* : a_m \dots a_0 \text{ is a path in } \mathfrak{A} \text{ for all } m \in \mathbb{N} \right\},$$

where $\mathfrak{A}$ is the modified prefix automaton of ζ . Since $|\lambda_j| < 1$ for $2 \leq j \leq r + s$, the set $\mathfrak{R}$ is well-defined. On Rauzy fractals and their relations to the Pisot conjecture, see [1, 12, 16, 21].

The *dual Rauzy fractal* is the set $\widehat{\mathfrak{R}} := \xi(\widehat{\mathcal{R}})$. We shall consider the dynamically defined subset of $\mathfrak{R}$,

$$\mathfrak{F} := \left\{ \sum_{i=0}^{\infty} (a_i \lambda^i)^* : a_0 a_1 \dots \in \mathcal{P} \right\}.$$

Some of the dynamical and geometrical properties of $\mathfrak{F}$, for the k -bonacci substitutions, i.e. $\zeta_{k,1}$, were studied in [18, 20]. The Rauzy fractal for $\zeta_{3,1}$ and the subset $\mathfrak{F}$ are shown in Figure 4. We call the set $\mathfrak{F}$ the *geometric canonical product* of the substitution ζ .

THEOREM 3.1. *The set $\mathfrak{F}$ is a symmetric subset of $\mathbb{R}^{k-1}$.*

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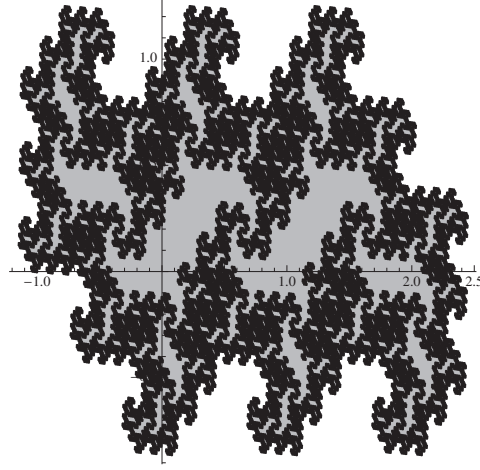


FIGURE 5. The Rauzy fractal of the substitution $\zeta_{3,2}$ and its symmetrical subset $\mathfrak{F}$ in black.

Proof. Let $a \in \mathcal{P}$, by definition $\hat{a} \in \mathcal{P}$. Then

$$\begin{aligned} \xi(\mathbf{a}) + \xi(\hat{\mathbf{a}}) &= \sum_{i=0}^{\infty} (a_i \lambda^i)^* + \sum_{i=0}^{\infty} (\hat{a}_i \lambda^i)^* = \sum_{i=0}^{\infty} (a_i \lambda^i)^* + \sum_{i=0}^{\infty} ((L - a_i) \lambda^i)^* \\ &= \sum_{i=0}^{\infty} (a_i \lambda^i)^* + \sum_{i=0}^{\infty} (L \lambda^i)^* - \sum_{i=0}^{\infty} (a_i \lambda^i)^* = L \sum_{i=0}^{\infty} (\lambda^i)^* \\ &= L \left(\frac{1}{1 - \lambda_2}, \dots, \frac{1}{1 - \lambda_r}, \frac{1}{1 - \lambda_{r+1}}, \dots, \frac{1}{1 - \lambda_{r+s}} \right), \end{aligned}$$

where $L/2$ is the center of symmetry of $l(\text{Pref})$. Therefore $\mathfrak{F}$ is symmetric and its center of symmetry is

$$\frac{L}{2} \left(\frac{1}{1 - \lambda_2}, \dots, \frac{1}{1 - \lambda_r}, \frac{1}{1 - \lambda_{r+1}}, \dots, \frac{1}{1 - \lambda_{r+s}} \right).$$

□

Now we will consider the following family of substitutions of type (1), given in the Introduction. These substitutions are unimodular Pisot (cf. [4]).

THEOREM 3.2. *Let $\mathfrak{R}$ be the Rauzy fractal of the substitutions of type (1) then $\hat{\mathfrak{R}} = \mathfrak{R}$.*

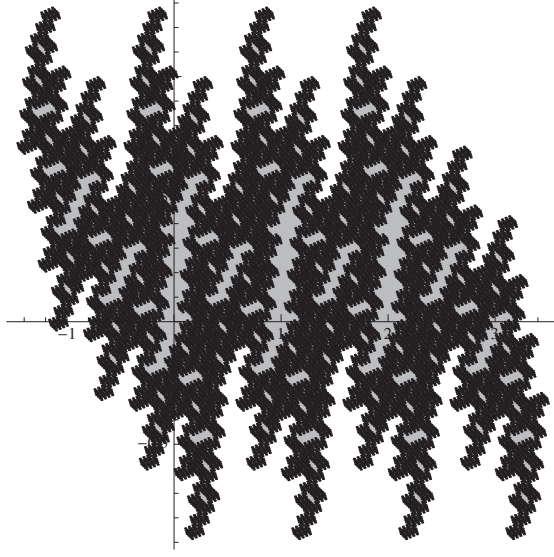


FIGURE 6. The Rauzy fractal of the substitution $\zeta_{3,3}$ and its symmetrical subset $\mathfrak{F}$ in black.

Proof. Since the characteristic polynomial of the incidence matrix is $x^k - nx^{k-1} - \dots - nx - 1$, we get

$$\xi(\bar{0}) = \xi(\mathbf{b}) = 0, \quad (2)$$

where $\bar{0} = 000\dots$ and $\mathbf{b} = \underbrace{1n\dots n}_{k-1}\bar{0} = 1\underbrace{n\dots n}_{k-1}0\underbrace{n\dots n}_{k-1}0\underbrace{n\dots n}_{k-1}0\dots$.

Let $e = e_m \dots e_0$ be a finite path in the automaton $\mathfrak{A}$. We shall associate $f = f_m \dots f_0$ a finite path in $\widehat{\mathfrak{A}}$ such that $\xi(e_0 \dots e_m \bar{0}) = \xi(f_0 \dots f_m \mathbf{b})$. This shall prove that $\mathfrak{R} \subset \widehat{\mathfrak{R}}$. The opposite inclusion is proved in a similar manner.

First we remark that if a is a finite path in $\mathfrak{A}$ then $a\bar{0}$ is an element of $\mathcal{R}$, since $0\dots 0$ is a path in $\mathfrak{A}$ starting and ending in the state 1. Let us recall that $\mathbf{a} = a_0 a_1 \dots$ is an element of $\mathcal{R}$, respectively $\widehat{\mathcal{R}}$, if and only if $a_m \dots a_0$ is a path in $\mathfrak{A}$, respectively $\widehat{\mathfrak{A}}$, for all $m > 0$. On the other hand $f_0 \dots f_m \mathbf{b}$ is an element of $\widehat{\mathcal{R}}$, if $f_m \dots f_0$ is a path on $\widehat{\mathfrak{A}}$: Let $v = v_0 \dots v_r$ be a finite prefix of $\mathbf{b}$, so it is a finite path on $\widehat{\mathfrak{A}}$ and the final state of the automaton $\widehat{\mathfrak{A}}$ when reads v is 1, so $v_r \dots v_0 f_m \dots f_0$ is a finite path in $\widehat{\mathfrak{A}}$.

If e is a path in $\widehat{\mathfrak{A}}$ then we say $f = e$. If e is not a path in $\widehat{\mathfrak{A}}$ then we can assume, without loss of generality, that $e_m \neq 0$. Let

$$d := \max_{0 \leq j \leq m} \{j : e_j \neq 0, e_{j-i} = 0 \text{ for } 1 \leq i \leq k\}.$$

Such d exists, since e is not a path in $\widehat{\mathfrak{A}}$. So we define

$$\begin{aligned} f_{d-k} &= 1; & f_{d-j} &= n, \text{ for } 1 \leq j \leq k-1; \\ f_d &= e_d - 1; & f_{d+i} &= e_{d+i}; \text{ for } 0 \leq i \leq m-d. \end{aligned}$$

We repeat this process with the word $e_0 \dots e_{d-k-1}$ until $d-k-1 \geq 0$. Due to the selection of f , it is a path in $\widehat{\mathfrak{A}}$. By the fact that λ_i 's are roots of the polynomial $x^k - nx^{k-1} - \dots - nx - 1$, we have:

$$\sum_{i=0}^m (e_i \lambda^i)^* = \sum_{i=0}^m (f_i \lambda^i)^*,$$

So $\xi(e_0 \dots e_m \overline{0}) = \xi(f_0 \dots f_m \mathbf{b})$. □

COROLLARY 3.3. *Let $\mathfrak{R}$ be the Rauzy fractal of a substitution of type (1) and $\mathfrak{F}$ its geometric canonical product. The sets $\mathfrak{R}$ and $\mathfrak{F}$ have the same center of symmetry. Moreover the center of symmetry is in $\mathfrak{R}$.*

Proof. Like in the proof of Theorem 3.1, Let $\mathbf{a} \in \mathcal{R}$ then

$$\begin{aligned} \xi(\mathbf{a}) + \xi(\hat{\mathbf{a}}) &= \sum_{i=0}^{\infty} (a_i \lambda^i)^* + \sum_{i=0}^{\infty} (\hat{a}_i \lambda^i)^* = \sum_{i=0}^{\infty} (a_i \lambda^i)^* + \sum_{i=0}^{\infty} ((n - a_i) \lambda^i)^* \\ &= \sum_{i=0}^{\infty} (n \lambda^i)^* = n \sum_{i=0}^{\infty} (\lambda^i)^*. \end{aligned}$$

Depending if k is even or odd we get

$$n \sum_{i=0}^{\infty} (\lambda^i)^* = \begin{cases} n \left(\frac{1}{1 - \lambda_2}, \dots, \frac{1}{1 - \lambda_{m-1}} \right), & \text{if } k = 2m; \\ n \left(\frac{1}{1 - \lambda_2}, \dots, \frac{1}{1 - \lambda_m} \right), & \text{if } k = 2m + 1. \end{cases}$$

Since the points $\xi(\mathbf{a})$ and $\xi(\hat{\mathbf{a}})$ are in $\mathfrak{R}$, the center of symmetry of this set is $(n/2) \sum_{i=0}^{\infty} (\lambda^i)^*$. Similarly if $a \in \mathcal{P}$ we get that $(n/2) \sum_{i=0}^{\infty} (\lambda^i)^*$ is the center of symmetry of $\mathfrak{F}$.

If $n = 2p$, then

$$\frac{n}{2} \sum_{i=0}^{\infty} (\lambda^i)^* = \sum_{i=0}^{\infty} (p \lambda^i)^* = \xi(\overline{p}).$$

It can be checked on the automaton $\mathfrak{A}$ that $\overline{p} = ppp \dots$ is in $\mathcal{R}$.

If $n = 2p + 1$, with $p \geq 0$. Using the relation $1 + n\lambda + \dots + n\lambda^{k-1} = \lambda^k$, we get:

$$\begin{aligned}
 \frac{n}{2} \sum_{i=0}^{\infty} (\lambda^i)^* &= \frac{1}{2} ((2p)^* + (1 + n\lambda + \dots + n\lambda^{k-1})^* + (n\lambda^k)^* + \\
 &\quad + (n\lambda^{k+1} + \dots + n\lambda^{2k})^* + (n\lambda^{2k+1})^* + \dots \\
 &= \frac{1}{2} ((2p)^* + ((2p+2)\lambda^k)^* + (2p\lambda^{k+1})^* + ((2p+2)\lambda^{2k+1})^* + \\
 &\quad + (2p\lambda^{2k+2})^* + \dots) \\
 &= \frac{1}{2} ((2p)^* + \sum_{i=0}^{\infty} (((2p+2)\lambda^{(k+1)i+k})^* + (2p\lambda^{(k+1)(i+1)}))^*) \\
 &= p^* + \sum_{i=0}^{\infty} (((p+1)\lambda^{(k+1)i+k})^* + (p\lambda^{(k+1)(i+1)}))^* \\
 &= \xi(\overbrace{p0 \dots 0(p+1)p}^{k-1}).
 \end{aligned}$$

Since $\overbrace{0 \dots 0(p+1)p}^{k-1}$ is in $\mathcal{R}$. We get that the center of symmetry of $\mathfrak{A}$ and $\mathfrak{F}$ is in $\mathfrak{A}$. □

4. Remarks

- (i) The involution $\Psi : W' \rightarrow W'$, $\mathbf{a} \mapsto \hat{\mathbf{a}}$, defined in Section 2, geometrically corresponds to the reflection through the center of symmetry of the set $\mathfrak{F}$, i.e.

$$\begin{aligned}
 \xi(\Psi(\mathbf{a})) &= \sum_{i \geq 0} (\hat{a}_i \lambda^i)^* = \sum_{i \geq 0} ((L - a_i) \lambda^i)^* \\
 &= \sum_{i \geq 0} (L \lambda^i)^* - \sum_{i \geq 0} (a_i \lambda^i)^* = L \sum_{i \geq 0} (\lambda^i)^* - \xi(\mathbf{a}).
 \end{aligned}$$

According to Theorem 3.1, the center of symmetry of the set $\mathfrak{F}$ is $(L/2) \sum_{i \geq 0} (\lambda^i)^*$.

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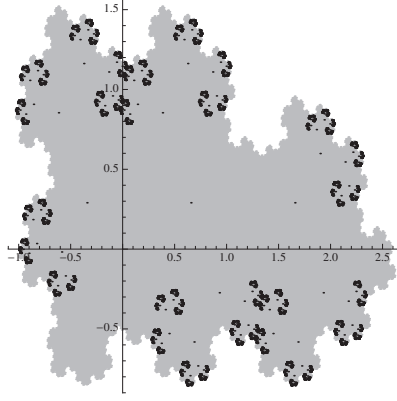


FIGURE 7. The Rauzy fractal of the substitution $1 \rightarrow 112$, $2 \rightarrow 13$, $3 \rightarrow 1$ and its symmetrical subset $\mathfrak{F}$ in black.

- (ii) It can be easily checked that the Rauzy fractals associated to the substitutions:

$$\zeta : \begin{cases} 1 \mapsto \underbrace{1 \cdots 1}_{n_1} 2 \\ 2 \mapsto \underbrace{1 \cdots 1}_{n_2} 3 \\ 3 \mapsto 1 \end{cases}$$

with $n_1 > n_2 \geq 1$ are not symmetric. In spite of the fact the corresponding sets $l(\text{Pref})$ are symmetric as subsets of $\mathbb{R}$.

- (iii) If $l(\text{Pref})$ is not symmetric it can be defined the canonical product in the following way: We consider $V \subset l(\text{Pref})$ so that it is the maximal symmetric subset of $l(\text{Pref})$. Let $\mathfrak{A}'$ be the automaton obtained from $\mathfrak{A}$, the modified prefix automaton of the substitution, where the transitions are those whose labels are in V . Later we take the connected component containing the initial state. To the automaton $\mathfrak{A}'$ defined in this way, we consider its dual $\widehat{\mathfrak{A}'}$ and its canonical product $\mathfrak{A}' \times \widehat{\mathfrak{A}'}$. If this automaton contains infinite paths we consider the set $\mathcal{P}'$ of paths in the canonical product. Let $\mathfrak{F}' = \xi(\mathcal{P}')$, i.e. the “geometrical realization” of the set $\mathcal{P}'$. The proof of Theorem 3.1 works for this set, so we have that $\mathfrak{F}'$ is symmetric as a subset of $\mathbb{R}^{k-1}$.

We show these constructions in the following example: Let us consider the substitution

$$1 \rightarrow 123, \quad 2 \rightarrow 12, \quad 3 \rightarrow 2.$$

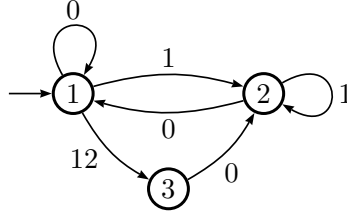
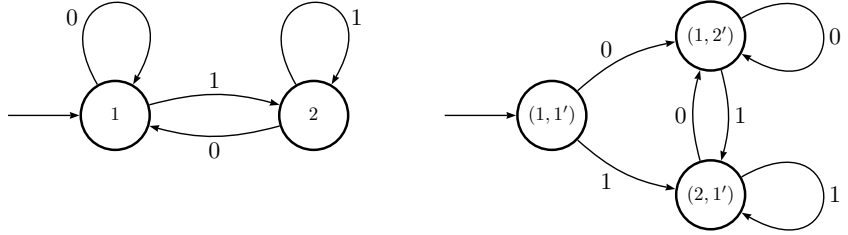


FIGURE 8. The prefix automaton of the Kolakoski substitution.


FIGURE 9. The automaton $\mathfrak{A}'$ (right) and its canonical product (left) associated to the Kolakoski substitution.

Which is known as the *Kolakoski substitution* (cf. [2]). Its prefix automaton, and the automata $\mathfrak{A}'$ and $\mathfrak{A}' \times \widehat{\mathfrak{A}'}$, are shown in Figures 8 and 9, respectively. In this case $l(\text{Pref}) = \{0, 1, \lambda - 1\}$, so we take $V = \{0, 1\}$, since it is symmetric and the respective automata are not trivial. Other selection of the set V ends in a trivial automaton. Let us remark that the automata $\mathfrak{A}'$ and $\mathfrak{A}' \times \widehat{\mathfrak{A}'}$ are equivalent, i.e. they generate the same sequences. In this case $\mathcal{P}' = \{0, 1\}^{\mathbb{N}}$. The Rauzy fractal of this substitution and its subset $\mathfrak{F}'$ can be seen in Figure 10. Let us point out that this Rauzy fractal is not symmetric.

- (iv) Let us denote by $\mathfrak{F}_{k,n}$ the geometric canonical product of the the substitution $\zeta_{k,n}$. Using the techniques in [18] or [17], we compute the Hausdorff dimension of the sets $\mathfrak{F}_{3,n}$, which is $2 \log(\rho) / \log(\lambda)$, where ρ is the Perron-Frobenius eigenvalue of the incidence matrix of the canonical product, i.e. $\mathfrak{A} \times \widehat{\mathfrak{A}}$. The characteristic polynomial of the incidence matrix of $\mathfrak{A} \times \widehat{\mathfrak{A}}$ is

$$(1 + x + x^2)(x^3 - nx^2 - nx + n - 1).$$

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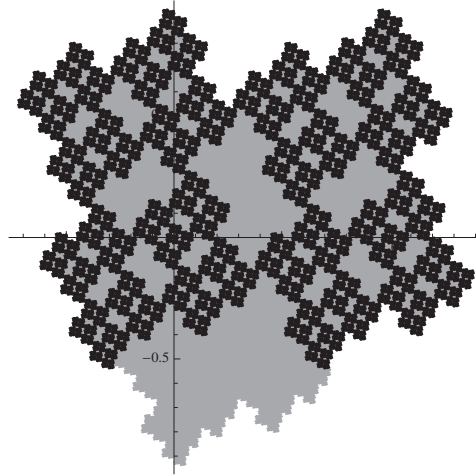


FIGURE 10. The Rauzy fractal of the Kolakoski substitution and its symmetrical subset $\mathfrak{F}$ in black.

So the Hausdorff dimensions of $\mathfrak{F}_{3,1}$, $\mathfrak{F}_{3,2}$, $\mathfrak{F}_{3,3}$, $\mathfrak{F}_{3,4}$ are approximately 1.579, 1.849, 1.929 and 1.9601, respectively. It can be proved using elementary arguments that $\text{HD}(\mathfrak{F}_{3,n}) < 2$, for $n \geq 1$, where HD stands for Hausdorff dimension. Numerical evidence suggests the fact $\text{HD}(\mathfrak{F}_{3,n+1}) > \text{HD}(\mathfrak{F}_{3,n})$.

In the Kolakoski substitution the Hausdorff dimension of the set $\mathfrak{F}$ is $2 \log(2) / \log(\lambda) \approx 1.752$, where λ is the real root of the polynomial $x^3 - 2x^2 - 1$.

- (v) We would like to remark, in the case where the embedding of the set of proper prefixes in $\mathbb{Q}(\lambda)$ is symmetric, if we consider the automaton of paths of length m , with $m > 1$, on the modified prefix automaton, we obtain the same set $\mathcal{P}$ as the maximal invariant subset of $\mathcal{R}$ under the involution Ψ . In fact: Let $\mathfrak{C}(m)$ be the automaton of paths of length m from the modified prefix automaton $\mathfrak{A}$. The set of infinite paths on $\mathfrak{C}(m)$ is the same set as the set of infinite paths on $\mathfrak{A}$, i.e. $\mathcal{R}$. The labels of $\mathfrak{C}(m)$ are words of length m in W' . So we can consider the dual automaton on $\mathfrak{C}(m)$ and get its canonical product. Let $\mathcal{P}(m)$ the set of infinite paths on the canonical product of $\mathfrak{C}(m)$. Since $\mathcal{P}(m)$ and $\mathcal{P}$ are, by construction, the maximal invariant subset of $\mathcal{R}$ under the involution Ψ , we get $\mathcal{P}(m) = \mathcal{P}$.

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