Uniform Distribution Theory **6** (2011), no.1, 21–31

uniform distribution theory

EQUIDISTRIBUTION IN THE d-DIMENSIONAL a-ADIC SOLENOIDS

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ABSTRACT. Given a probability measure μ on the d-dimensional a-adic solenoid Ω_a^d and an endomorphism T of Ω_a^d, we consider the relation between uniform distribution of the sequence $T^n\mathbf{x}$ for μ-almost all $\mathbf{x} \in \Omega_a^d$ and the behavior of μ relative to the translations by some rational subgroups of Ω_a^d. The main result of this note is an extension of the corresponding result for the d-dimensional torus $\mathbb{T}^d$ due to B. Host [6].

Communicated by Radhakrishnan Nair

1. Introduction

Given a probability measure μ on the d-dimensional torus $\mathbb{T}^d$ and an endomorphism T of $\mathbb{T}^d$, B. Host considered the relation between uniform distribution of the sequence $T^n t$ for μ-almost all $t \in \mathbb{T}^d$ and the behavior of μ relative to the translations by some rational subgroups of $\mathbb{T}^d$. The main aim of this note is to extend Host's theorems (see [6, Therem 1 and Theorem 2]) and their proofs to the d-dimensional a-adic solenoid. The d-dimensional a-adic solenoid is a compact group which can be considered as an generalization of $\mathbb{T}^d$ (see [1, 4]).

By $\mathbb{P} \subset \mathbb{N}$ we denote the set of primes. For a prime number $p \in \mathbb{P}$ let $\mathbb{Q}_p$ ($\mathbb{Z}_p$, respectively) denote the p-adic field of rational numbers (the ring of p-adic integers, respectively) with the p-adic norm $|\cdot|_p$. We write $\mathbb{Q}_\infty$ for $\mathbb{R}$ and $|\cdot|_\infty$ for the usual absolute value. Let $a \in \mathbb{N}$ be a square-free number, that is a is a product of different prime numbers, i.e., $a = p_1 p_2 \dots p_s$. For any positive integer d let Ω_a^d be a quotient group of the additive group $\mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \dots \times \mathbb{Q}_{p_s}^d$

2010 Mathematics Subject Classification: 11J71, 11K06, 54H20.
Keywords: Uniform distribution modulo 1, equidistribution in probability, a-adic solenoid, topological dynamics.
Research was supported in part by the MNiSW research grant N N201 393937.

by a discrete subgroup

$$B^d = \Big\{(b, -b, \dots, -b) : b \in \mathbb{Z}[1/a]^d\Big\},$$

where $\mathbb{Z}[1/a]$ is the ring obtained from $\mathbb{Z}$ by adjoining $1/a$. Thus,

$$\Omega_a^d = \mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \cdots \times \mathbb{Q}_{p_s}^d \Big/ B^d.$$

The quotient group Ω_a^d is a compact Abelian group called the d-dimensional *a-adic solenoid* (see [4]).

The ring $\mathrm{End}(\Omega_a^d)$ of continuous endomorphisms of Ω_a^d is isomorphic to $\mathrm{M}(d, \mathbb{Z}[1/a])$, where $\mathrm{M}(d, R)$ denotes the ring of $d \times d$ matrices over a ring R. The action of the matrix $C \in \mathrm{M}(d, \mathbb{Z}[1/a])$ on $\prod_{j=0}^s \mathbb{Q}_{p_j}^d$ is given by

$$C(x_0, x_1, \dots, x_s) = (Cx_0, Cx_1, \dots, Cx_s),$$

$(x_0, x_1, \dots, x_s) \in \prod_{j=0}^s \mathbb{Q}_{p_j}^d$, and the vectors $x_k \in \mathbb{Q}_{p_k}^d$, $k = 0, \dots, s$, are considered as column vectors.

If C is an endomorphism of Ω_a^d, then the dual endomorphism $\hat{C}$ is given by the same matrix acting from the right on $\mathbb{Z}[1/a]^d$.

Let $T \in \mathrm{M}(d, \mathbb{Z}[1/a])$. According to [5, 6], we say that the sequence $T^n\mathbf{x}$, $\mathbf{x} \in \Omega_a^d$ is *equidistributed in probability for the measure* μ if, for every weak-$*$ neighborhood U of the Lebesgue measure,

$$\lim_{N\to\infty} \mu \left\{ \mathbf{x} \in \Omega_a^d : \frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n\mathbf{x}} \notin U \right\} = 0.$$

For integer $q > 1$, define the following subgroup D_q of Ω_a^d,

$$D_q = \left\{ \left(\frac{j_1}{q^n}, \dots, \frac{j_d}{q^n}, \frac{-j_1}{q^n}, \dots, \frac{-j_d}{q^n}, \dots, \frac{-j_1}{q^n}, \dots, \frac{-j_d}{q^n} \right) + B^d : 0 \le j_1, \dots, j_d \le q^n, \;\; n \ge 1 \right\}. \tag{1.1}$$

Let

$$D_{q,k} = \left\{ \left(\frac{j_1}{q^k}, \dots, \frac{j_d}{q^k}, \frac{-j_1}{q^k}, \dots, \frac{-j_d}{q^k}, \dots, \frac{-j_1}{q^k}, \dots, \frac{-j_d}{q^k} \right) + B^d : 0 \le j_1, \dots, j_d \le q^k \right\}. \tag{1.2}$$

Then

$$D_q = \bigcup_{k \ge 1} D_{q,k}.$$

Define the following measures

$$\omega_k = \sum_{\mathbf{x}\in D_{q,k}} \delta_{\mathbf{x}} * \mu.$$

Let

$$\varphi_k(\mathbf{x}) = \frac{d\mu(\mathbf{x})}{d\omega_k(\mathbf{x})} \tag{1.3}$$

be the Radon-Nikodym derivative.

We say that the probability measure μ on Ω_a^d is *D_q-conservative* if for every Borel set E with $\mu(E) > 0$, there exists $\mathbf{y} \in D_q$, $\mathbf{y} \neq 0$, with $\mu(E \cap (\mathbf{y} + E)) > 0$.

We say that μ is *D_q-conservative with exponential decay* if

$$\liminf_{k\to\infty} -\frac{1}{k} \log \varphi_k(\mathbf{x}) > 0, \quad \mu\text{-a.e.}$$

The main result of this note is the following.

Theorem 1.4. *Let $T \in \mathrm{M}(d, \mathbb{Z}[1/a])$, where $a = p_1 p_2 \ldots p_s$ is a product of different primes. Let D_q be the subgroup of Ω_a^d, defined in* (1.1), *with $q = q_1^{\alpha_1} \ldots q_m^{\alpha_m} > 1$, where $q_i \in \mathbb{P}$, $\alpha_i \geq 1$.*

Assume that

- (i) *for every integer $r > 1$ the characteristic polynomial of T^r is irreducible over $\mathbb{Q}$,*
- (ii) *for $j = 1, \ldots, s$, $|q|_{p_j} = 1$,*
- (iii) *for $j = 1, \ldots, m$, $|\det T|_{q_j} = 1$.*

Then

1. *if the probability measure μ on Ω_a^d is D_q-conservative, then the sequence $T^n\mathbf{x}$ is equidistributed in probability for μ;*
2. *if the probability measure μ on Ω_a^d is D_q-conservative, with exponential decay then for μ-a.e. $\mathbf{x} \in \Omega_a^d$ the sequence $T^n\mathbf{x}$ is equidistributed.*

Remark. Condition (i) implies that $\det T \neq 0$, that is $T \in \mathrm{GL}(d, \mathbb{R})$.

Remark. It should be emphasized that when an appropriate formulation, given in Theorem 1.4 above, of the extension of Host's results [6, Theorem 1 and Theorem 2] is found then the proof is relatively easy. It amounts to some modifications of the original proofs which are necessary in the new more general setting.

2. Lemmas

For a given positive integer q we denote by $\Omega_a^d(q)$ the subgroup of Ω_a^d consisting of all elements whose order divides q.

Lemma 2.1. *For every $q \in \mathbb{N}$, the subgroup*

$$\Omega_a^d(q) \simeq \mathbb{Z}[1/a]^d/q\mathbb{Z}[1/a]^d \simeq \big(\mathbb{Z}[1/a]/q\mathbb{Z}[1/a]\big)^d$$

is finite.

Proof. (See [1] Lemma II.13) It is enough to show that $\mathbb{Z}[1/a]^d/q\mathbb{Z}[1/a]^d$ is a finite group. Let $\gamma_1,\ldots,\gamma_c \in \mathbb{Z}[1/a]^d$ be different elements of $\mathbb{Z}[1/a]^d/q\mathbb{Z}[1/a]^d$. Consider for $n = 0,1,2,\ldots$ the following subgroups

$$G_n = \mathbb{Z}[1/a]^d \cap \frac{1}{n!}\mathbb{Z}^d \quad \text{of} \quad \mathbb{Z}[1/a]^d.$$

Clearly, G_n is a rank d subgroup of $\frac{1}{n!}\mathbb{Z}^d$ and so $G_n \simeq \mathbb{Z}^d$. Since $G_n \subset G_{n+1}$ and $\bigcup_{n=0}^{\infty} G_n = \Omega_a^d$, there exists $k \in \mathbb{N}$ such that for every $1 \le i \le c$, we have $\gamma_i \in G_k$. The elements $\gamma_1,\ldots,\gamma_c$ belong to different classes modulo qG_k. Since $\mathbb{Z}^d/q\mathbb{Z}^d$ has order q^d we get $c \le q^d$. $\square$

Remark. It was pointed out by the referee that there is a simple formula for the cardinality c of the group $\mathbb{Z}[1/a]^d/q\mathbb{Z}[1/a]^d$. More specifically, the cardinality of $\mathbb{Z}[1/a]^d/q\mathbb{Z}[1/a]^d$ is the number of points fixed by the endomorphism $x \mapsto (1-q)x$ on $\mathbb{Z}[1/a]^d$. There is a well-known formula for the cardinality of the later set, given for example in [2, Lemma 5.2], from which it follows that

$$c = \left(|q|_\infty \prod_{p|a} |q|_p\right)^d \le \ q^d.$$

This formula can be view via an adelic covering lemma that makes this just a volume calculation in some finite product of p-adic fields (see [2]).

Let a be a product of different prime numbers $a = p_1p_2\ldots p_s$, $p_j \in \mathbb{P}$, and let $T = (t_{ij}) \in \mathrm{M}(d,\mathbb{Z}[1/a]) \cap \mathrm{GL}(d,\mathbb{R})$. Set

$$r = 2 + d(d-1)/2, \tag{2.2}$$

and consider, for an integer q satisfying (ii) of Theorem 1.4, the matrix

$$\tilde{T} \in \mathrm{M}\big(d,\mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]\big),$$

with entries

$$\tilde{t}_{ij} = t_{ij} \bmod q^r\mathbb{Z}[1/a] = t_{ij} + q^r\mathbb{Z}[1/a].$$

Lemma 2.3. *Let $T \in \mathrm{M}(d, \mathbb{Z}[1/a]) \cap \mathrm{GL}(d, \mathbb{R})$, $r \geq 1$. Let $q = q_1^{\alpha_1} q_2^{\alpha_2} \dots\ q_m^{\alpha_m}$, $q_i \in \mathbb{P}$, $\alpha_i \in \mathbb{N}$. Assume that for $1 \leq i \leq m$, $|\det T|_{q_i} = 1$. Then there exists a number $\tau \in \mathbb{N}$ such that*

$$T^\tau \equiv I_d \bmod q^r \mathbb{Z}[1/a]^d,$$

where I_d stands for the identity $d \times d$-matrix.

P r o o f. By Lemma 2.1 the matrix $\tilde{T}$ acts naturally on the finite module

$$\big(\mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]\big)^d \ \text{ over the finite ring } \ \mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a].$$

Thus we have an action of the semigroup $\mathbb{N}$ on $\big(\mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]\big)^d$, given by $k.x = T^k x$, $k \in \mathbb{N}$, $x \in \big(\mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]\big)^d$. Since $|\det T|_{q_i} = 1$, for $1 \leq i \leq m$, we conclude that $\det \tilde{T}$ is invertible in $\mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]$, hence

$$\tilde{T} \in \mathrm{GL}\big(d, \mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]\big).$$

Thus $\{\tilde{T}^k : k \in \mathbb{N}\}$ is a semigroup contained in the finite group

$$\mathrm{GL}\big(d, \mathbb{Z}[1/a]/q^r\mathbb{Z}[1/a]\big);$$

it follows that $\{\tilde{T}^k : k \in \mathbb{N}\}$ is a group. Thus there exists a τ such that $\tilde{T}^\tau = I_d$, and the lemma is proved. □

Denote

$$I(N) = \{0, 1, \dots, N-1\}^d.$$

Let us fix some $\varepsilon \in (0,1)$, and let α be an integer so large that the set

$$\Lambda = \big\{n \in \mathbb{N}^d : n_i \neq n_j \bmod p^\alpha \text{ for all } i \neq j \text{ and all prime divisors } p \text{ of } q\big\} \tag{2.4}$$

satisfies

$$\mathrm{Card}(I(N) \cap \Lambda) \geq (1 - \varepsilon^d) N^d \ \text{ for all } N \text{ large enough}.$$

Let $p \in \mathbb{P} \cup \{\infty\}$, and let $A = (a_{ij}) \in \mathrm{M}(d, \mathbb{Q}_p)$ and $x = (x_1, \dots, x_d)^t \in \mathbb{Q}_p^d$ be a column vector. Here and in what follows all vectors are column vectors unless explicitly written as transposed. We define the norms of A and x by

$$\|A\|_p = \max_{i,j} |a_{ij}|_p \quad \text{and} \quad \|x\|_p = \max_j |x_j|_p.$$

Given $q \in \mathbb{Z}$, $q \geq 2$, and $m, n \in \mathbb{Z}^d$, we write $m = n \mod q$ if for every $1 \leq j \leq d$, $m_j = n_j \bmod q$.

Let $A \in \mathrm{M}(d, \mathbb{Z}_p)$ and $\|I_d - A\|_p \leq p^{-1}$. It is known (see e.g., [3], [8]) that the following series

$$\log A := \sum_{n=1}^{\infty} -\frac{1}{n}(I_d - A)^n$$

converges in $\mathrm{M}(d,\mathbb{Q}_p)$ and $\log A \in \mathrm{M}(d,\mathbb{Z}_p)$. Moreover, if $A \in \mathrm{M}(d,\mathbb{Z}_p)$ and $\|A\|_p \le p^{-2}$, then one can define $\exp A$ as a series

$$\exp A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

converging in $\mathrm{M}(d,\mathbb{Q}_p)$, and one has $\exp A \in \mathrm{M}(d,\mathbb{Z}_p)$.

Let M be the transpose matrix of T^τ, where τ is as in Lemma 2.3, that is $M = (T^\tau)^t$. Now we are able to generalize the proof of the fundamental bound from [6, § 4] to our setting and get the following result.

Lemma 2.5. *Under the assumptions of Theorem 1.4 there exists an integer $l > 0$ such that for $k \ge l$, $m, n \in \Lambda$, and $b \in \mathbb{Z}[1/a]^d$ if $m = n \mod q^l$ and $\sum_{i=1}^d M^{m_i} b = \sum_{i=1}^d M^{n_i} b \mod q^{l+k}\mathbb{Z}[1/a]$, then $m = n \mod q^k$.*

P r o o f. We follow [6]. By Lemma 2.3 each entry of the matrix $I_d - T^\tau$ is equal to 0 modulo $q^r\mathbb{Z}[1/a]$. Thus, also $(I_d - M)_{ij} = 0 \mod q^r\mathbb{Z}[1/a]$. Hence, the ijth entry of the matrix $I_d - M$ belongs to $q^r\mathbb{Z}[1/a]$, i.e., is equal to $q^r \frac{k_{ij}}{p_1^{\alpha_{1ij}} \dots p_s^{\alpha_{sij}}}$ for some k_{ij} and $\alpha_{1ij}, \dots, \alpha_{sij} \in \mathbb{Z}$. Using the assumption (ii) of Theorem 1.4, for every prime divisor p of q we have

$$\begin{aligned} \|I_d - M\|_p &= \max_{i,j} |(I_d - M)_{ij}|_p = \max_{i,j} \left| q^r \frac{k_{ij}}{p_1^{\alpha_{1ij}} \dots p_s^{\alpha_{sij}}} \right|_p \\ &= p^{-r}|k_{ij}|_p \le p^{-r} \le p^{-2}. \end{aligned}$$

Hence, the following matrices are well defined

$$A = p^{-r}\log(M) \in \mathrm{M}(d,\mathbb{Z}_p)$$

and

$$M^x := \exp(xp^r A) \in \mathrm{M}(d,\mathbb{Z}_p) \text{ for } x \in \mathbb{Z}_p.$$

For a given non-zero element $b \in \mathbb{Z}[1/a]^d$ we define

$$F_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d, \qquad F_p(x) = \sum_{i=1}^d M^{x_i} b.$$

Let

$$V(x) = \prod_{1\le i<j\le d} (x_j - x_i) \quad \text{and} \quad \delta_p = \left| D \det(A) \prod_{i=0}^{d-1} \frac{p^{ri}}{i!} \right|_p,$$

where $D \in \mathbb{Q}_p$ is the determinant of the vectors $b, Ab, \dots, A^{d-1}b$ in $\mathbb{Q}_p^d$. By [6, Lemma 1], $D \ne 0$ (here the assumption (i) of Theorem 1.4 is used).

We need the following lemma.

Lemma 2.6 ([6, Lemma 3]). *Let $x \in \mathbb{Z}_p^d$ and $x_i \neq x_j$ for $i \neq j$. Then for all $y \in \mathbb{Z}_p^d$ such that $\|y\|_p < p^{r-2}\delta_p |V(x)|_p$ we have*

$$\|F_p(x+y) - F_p(x)\|_p \geq p^{-r}\delta_p |V(x)|_p \|y\|_p.$$

Now we proceed as in [6, Section 4.3]. We have $F_p(n) \in \mathbb{Z}[1/a]^d$ for $n \in \mathbb{N}^d$. We have $|V(n)|_p \geq p^{-d(d-1)\alpha/2}$ for all $n \in \Lambda$ and every prime divisor p of q ($p = q_i$) by (2.4). We take $\beta > 0$ such that $\beta \geq 2 - r - \frac{\log \delta_p}{\log p} + \frac{d(d-1)}{2}\alpha$ for all $p = q_i$, $i = 1, \dots, m$. It follows from Lemma 2.6 that for all $p = q_i, m, n \in \Lambda$ and $\|m - n\|_p \leq p^{-\beta}$ imply

$$\|F_p(m) - F_p(n)\|_p \geq p^{-\beta - 2r + 2}\|m - n\|_p.$$

Notice that $m = n \mod q^k$ means that $\|m - n\|_p \leq p^{-k}$. Similarly, using (ii) of Theorem 1.4 we see that the condition

$$\sum_{i=1}^{d} M^{m_i} b = \sum_{i=1}^{d} M^{n_i} b \bmod q^{l+k}\mathbb{Z}[1/a]$$

means that

$$\|F_p(m) - F_p(n)\|_p \leq p^{-l-k}.$$

Hence, Lemma 2.5 follows. □

3. Proof of Theorem 1.4

Every $x \in \mathbb{Q}_p$ can be uniquely expressed as a convergent, in $|\cdot|_p$-norm, sum (*Hensel representation*),

$$x = \sum_{k=t}^{\infty} x_k p^k, \quad \text{for some} \quad t \in \mathbb{Z} \quad \text{and} \quad x_k \in \{0, 1, \dots, p-1\}.$$

The *fractional part* of $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$ is 0 if the number t in the Hensel representation is greater than or equal to 0, and equal to $\sum_{k<0} x_k p^k$, if $t < 0$. We write $\{x\}$ for the usual fractional part of $x \in \mathbb{R}$.

Recall that, for $p \in \mathbb{P} \cup \{\infty\}$, $\hat{\mathbb{Q}}_p$ is topologically isomorphic with $\mathbb{Q}_p$ and the action of the character $\chi_x \in \hat{\mathbb{Q}}_p$ corresponding to $x \in \mathbb{Q}_p$ is

$$\chi_x(y) = \exp\bigl(2\pi i \{xy\}_p\bigr).$$

We use the notation $p_0 = \infty$. Let $S = \{p_0, p_1, \ldots, p_s\}$. Denote by $\mathbb{Q}_S^d = \prod_{j=0}^s \mathbb{Q}_{p_j}^d$ the "covering space" of Ω_a^d. Since $\hat{\Omega}_a^d = \mathbb{Z}[1/a]^d$ the characters of Ω_a^d are indexed by vectors $b \in \mathbb{Z}[1/a]^d$ and are of the form

$$\chi_b(\mathbf{x}+B) = \prod_{j=0}^{s} e^{2\pi i \{\langle b, x_j\rangle\}_{p_j}}, \quad \mathbf{x} = (x_0, \ldots, x_s) \in \mathbb{Q}_S^d.$$

For a given non-zero $b \in \mathbb{Z}[1/a]^d$, let

$$S_N(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_b(T^n \mathbf{x})$$

and

$$S_N^\tau(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_b(T^{n\tau}\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_{M^n b}(\mathbf{x}),$$

where M is the transpose matrix of T^τ and τ is as in Lemma 2.3.

We have

$$\left(S_N^\tau(\mathbf{x})\right)^d = \frac{1}{N^d}\sum_{n\in I(N)} \chi_{\sum_{j=1}^d M^{n_j} b}(\mathbf{x}),$$

where $I(N) = \{0, 1, \ldots, N-1\}^d$. Let

$$\tilde{S}_N^\tau(\mathbf{x}) = \frac{1}{N^d}\sum_{n\in I(N)\cap\Lambda} \chi_{\sum_{j=1}^d M^{n_j} b}(\mathbf{x}),$$

where Λ is defined in (2.4). Then, for N large enough,

$$\left|(S_N^\tau(\mathbf{x}))^d - \tilde{S}_N^\tau(\mathbf{x})\right| \le \varepsilon^d. \tag{3.1}$$

Lemma 3.1. *There exists a constant $C > 0$ such that for all $k \ge 2l$, where l is from Lemma 2.5, and for all $N \ge q^k$,*

$$\int_{\Omega_a^d} \frac{\left|\tilde{S}_N^\tau(\mathbf{x})\right|^2}{\varphi_k(\mathbf{x})} d\mu(\mathbf{x}) \le C,$$

where φ_k is defined in (1.3).

P r o o f. We note that $\mathrm{card}(D_{q,k}) = q^{dk}$. Using the orthogonality of characters, i.e., the fact that for every non-zero element $b \in \mathbb{Z}[1/a]^d$, $\sum_{\mathbf{x}\in D_{q,k}} \chi_b(\mathbf{x}) = 0$, we get, in the same way as in [6, § 2.3], the following estimate

$$\int_{\Omega_a^d} \frac{\left|\tilde{S}_N^\tau(\mathbf{x})\right|^2}{\varphi_k(\mathbf{x})} d\mu(\mathbf{x}) \leq q^{dk} \sum_{\mathbf{j}\in(\mathbb{Z}[1/a]/q^k\mathbb{Z}[1/a])^d} \left(\frac{1}{N^d}\operatorname{card}\left\{n \in I(N)\cap\Lambda : \sum_{i=1}^d M^{n_j}b = \mathbf{j} \bmod q^k\mathbb{Z}[1/a]\right\}\right)^2$$

which together with Lemma 2.5 gives, as in [6], the required bound with

$$C = (q^{2l} + q^l)^{2d}. \qquad \square$$

P r o o f o f T h e o r e m 1.4 (1). By the classical results on uniformly distributed sequences in compact groups [7] we have to show that for every non-zero

$$b \in \mathbb{Z}[1/a]^d, \ \lim_{N\to\infty} S_N(\mathbf{x}) = 0 \text{ in } \mu - \text{probability}.$$

As in [6] it is enough to prove that

$$\lim_{N\to\infty} S_N^\tau(\mathbf{x}) = 0 \text{ in } \mu - \text{probability}.$$

Lemma 3.2. *A probability measure μ on Ω_a^d is D_q-conservative if and only if*

$$\varphi_k(\mathbf{x}) \to 0 \ \mu - a.e. \text{ as } \ k \ \text{ tends to} + \infty.$$

P r o o f. It is the same as the proof of the corresponding result for the 1-dimensional torus [5, Lemma 2]. $\square$

Now we proceed as in [6]. By Lemma 3.2 for every $\varepsilon > 0$, there exists a Borel subset

$$E \subset \Omega_a^d \quad \text{with} \quad \mu(E) > 1 - \varepsilon \quad \text{and} \quad k > 0$$

such that

$$\varphi_k(\mathbf{x}) < \varepsilon^{2d+1} \quad \text{for all } \ \mathbf{x} \in E.$$

By Lemma 3.1 we have, for N sufficiently large,

$$\int_E \left|\tilde{S}_N^\tau(\mathbf{x})\right|^2 d\mu(\mathbf{x}) \leq \varepsilon^{2d+1} \int_E \frac{\left|\tilde{S}_N^\tau(\mathbf{x})\right|^2}{\varphi_k(\mathbf{x})} d\mu(\mathbf{x}) \leq \varepsilon^{2d+1} C.$$

Hence, by (3.1),

$$\begin{aligned} \mu\{\mathbf{x} : \left|S_N^\tau(\mathbf{x})\right| \geq 2\varepsilon\} &\leq \mu\{\mathbf{x} : \left|\tilde{S}_N^\tau(\mathbf{x})\right| \geq \varepsilon^d\} \\ &\leq \varepsilon + \varepsilon^{-2d} \int_E \left|\tilde{S}_N^\tau(\mathbf{x})\right|^2 d\mu(\mathbf{x}) \\ &\leq (1 + C)\varepsilon, \end{aligned}$$

for N sufficiently large, and part (1) of Theorem 1.4 is proved. $\square$

Proof of Theorem 1.4(2). We have to show that $\lim_{N\to\infty} S_N^\tau(\mathbf{x}) = 0$ for μ-a.e. $\mathbf{x}$. The proof given in [6] works in this case again. We include here the main steps for the convenience of the reader.

The measure μ is D_q-conservative with exponential decay. Hence, for every $\varepsilon > 0$, we can find $\eta > 0$ and the set F with $\mu(F) > 1 - \frac{\varepsilon}{2}$, such that

$$\liminf_{k\to\infty} -\frac{1}{k}\log\varphi_k(\mathbf{x}) > \eta \qquad \text{for } \mathbf{x} \in F.$$

Hence, there is a set E with $\mu(E) > 1 - \varepsilon$ and $K \in \mathbb{N}$, $K \geq 2l$, where l is from Lemma 2.5, such that

$$\varphi_k(\mathbf{x}) < e^{-k\eta} \qquad \text{for } \mathbf{x} \in E \quad \text{and} \quad k \geq K.$$

Using Lemma 3.1, similarly as in the proof of part (1) above, we get

$$\int_E \left|\tilde{S}_N(\mathbf{x})\right|^2 \leq Ce^{-k\eta} \qquad \text{for } k \geq K \quad \text{and} \quad N \geq q^k,$$

and consequently, taking $k = [\log N/\log q]$,

$$\int_E \left|\tilde{S}_N(\mathbf{x})\right|^2 \leq Ce^{\eta}N^{-\eta/\log q} \text{ for } N \text{ sufficiently large.}$$

This shows that if $m\eta/\log q > 1$, then $\lim_{N\to\infty} \tilde{S}_{N^m} = 0$ a.e. on E. This implies, in a standard way, that for μ-a.e. $\mathbf{x} \in E$, $\limsup_{N\to\infty} |S_N(\mathbf{x})| \leq \varepsilon$, and the result follows. □

Acknowledgements. The author wishes to thank an anonymous referee for several remarks that improved the overall presentation of the result. In particular, for bringing to our attention reference [2].

REFERENCES

[1] BEREND, D.: *Multi-invariant sets on compact abelian groups*, Trans. Amer. Math. Soc. **286** (1984), no. 2, 505–535.

[2] CHOTHI, V. – EVEREST, G. – WARD. T.: *S-integer dynamical systems: periodic points*, J. Reine Angew. Math. **489** (1997), 99–132.

[3] GOUVÊA, F.Q.: *p-adic Numbers. An Introduction* (2nd edition). Universitext, Springer-Verlag, Berlin, 1997.

[4] HEWITT, E. – ROSS. K. A.: *Abstract Harmonic Analysis. Volume I. Structure of Topological Groups, Integration Theory, Group Representations*, (2nd edition). Fundamental Principles of Mathematical Sciences **115**, Springer, Berlin, 1979.

[5] HOST, B.: *Nombres normaux, entropie, translations*, Israel J. Math. **91** (1995), no. 1–3, 419–428.

[6] HOST, B.: *Some results of uniform distribution in the multidimensional torus*, Ergodic Theory Dynam. Systems **20**, (2000) no. 2, 439–452.

[7] KUIPERS, L. – NIEDERREITER, H.: *Uniform Distribution of Sequences*. In: *Pure and Applied Mathematics*, Wiley-Interscience, John Wiley & Sons, New York--London-Sydney, 1974.

[8] ROBERT, A. M.: *A Course in p-adic Analysis. Graduate Texts in Mathematics* **198**, Springer-Verlag, New York, 2000.

Received Septemer 23, 2010
Accepted October 10, 2010

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