Uniform Distribution Theory **4** (2009), no.2, 133–151

uniform distribution theory

A NOTE ON WARING'S PROBLEM WITH CONVERGENT SIEVE SEQUENCES

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ABSTRACT. Let $\mathcal{V}$ be a set of pairwise coprime integers not containing 1, and we assume that there is some $\delta > 0$ such that $\sum_{v\in\mathcal{V}} v^{-1+\delta}$ is finite. An asymptotic formula for Waring's Problem where the variables are assumed to be not divisible by any $v \in \mathcal{V}$ is proved.

Communicated by Robert F. Tichy

1. Introduction

Waring's Problem with multiplicatively restricted variables has been widely studied. For instance, Esterman [7] investigated if an integer can be represented as a sum of squares of squarefree integers; Baker and Brüdern [2] proved that almost all integers can be represented as a sum of four cubes of squarefree integers; among others, Waring's Problem with smooth numbers has been explored by Balog and Sárközy [3] and Harcos [8]; Brüdern and Fouvry [4] showed Lagrange's Four Square Theorem where the variables are almost prime. In this paper, we are interested in Waring's Problem where the variables are 'sieved' by integers of a certain set $\mathcal{V}$.

Let $\mathcal{V}$ be a set of pairwise coprime integers not containing 1. Throughout this paper, we assume that there is some $\delta > 0$ such that

$$\sum_{v\in\mathcal{V}} \frac{1}{v^{1-\delta}} \tag{1}$$

2000 Mathematics Subject Classification: 11P05, 11N25.
Keywords: Waring's Problem, sieve sequences, B-free numbers.
The author was supported by the Austrian Science Foundation (FWF), projects S9610 and S9611, and wants to thank J. Brüdern, J. Thuswaldner, and R. Tichy for helpful discussions.

converges. Introduce

$$\chi_{\mathcal{V}}(n) := \begin{cases} 1, & \text{if } v \nmid n \text{ for all } v \in \mathcal{V}; \\ 0\,, & \text{otherwise.} \end{cases}$$

The notion of the set $\{n \in \mathbb{N} : \chi_{\mathcal{V}}(n) = 1\}$ has been introduced by Erdös [6]. The distribution of this set in residue classes has been studied by Jancevskis [9]. Alkan and Zaharescu [1] investigated the problem of finding integers n in short arithmetic progressions such that $\chi_{\mathcal{V}}(n) = 1$. An integer n satisfying $\chi_{\mathcal{V}}(n) = 1$ is also called a $\mathcal{V}$-free number (e.g. in [1]).

One of the most prominet examples of such a sieve sequence is the set of squarefree numbers. In this case $\mathcal{V} = \{p^2 : p \text{ prime }\}$ and we have $\chi_{\mathcal{V}} = \mu^2$. Now, for any $\varepsilon > 0$ and $\delta = 1/2 - \varepsilon$ our assuption (1) holds.

Let $s, k \in \mathbb{N}$. We denote by $R_{\mathcal{V}}(N)$ the number of solutions of

$$x_1^k + \ldots + x_s^k = N \tag{2}$$

with $\chi_{\mathcal{V}}(x_1) = \ldots = \chi_{\mathcal{V}}(x_s) = 1$.

In this paper, the number s of summands is only of secondary interest. Thus we assume for the sake of simplicity that $s > 2^k$. This bound is not best possible and the techniques of the present paper yield also results for smaller s. The range of s depends on the current state of results improving Hua's Lemma (for a survey, see e.g., [14]). Here, we only make use of the original form of Hua's Lemma (see e.g., [13, Lemma 2.5]).

Theorem 1. *Let $\mathcal{V}$ be a set of pairwise coprime integers not containing 1 and assume that there is some $\delta > 0$ such that (1) is finite. Then there is some $\rho > 0$ such that*

$$R_{\mathcal{V}}(N) = J\mathfrak{S}_{\mathcal{V}}(N)\, N^{s/k-1} + O\left(N^{s/k-1-\rho}\right),$$

where $J > 0$ is defined at the end of Section 3.2 below and only depends on s, k. The arithmetic function $\mathfrak{S}_{\mathcal{V}}(N)$, called the singular series, is defined in Lemma 3. We have $\mathfrak{S}_{\mathcal{V}}(N) \ll 1$ for all $N \in \mathbb{N}$. The implicit O–constant depends on s, k, δ, and ρ only.

Unfortunately, we could not determine whether $\mathfrak{S}_{\mathcal{V}}(N) > 0$ for an arbitrary set $\mathcal{V}$ of pairwise coprime integers. The difficulty lies in the fact that the function D associated with the singular series $\mathfrak{S}_{\mathcal{V}}(N) = \sum_{q \geq 1} D(q)$ might not be multiplicative if $\mathcal{V}$ is not a set of prime powers (see Remark 1 after Lemma 4).

However, if we restrict $\mathcal{V}$ to be a set of prime powers, we are able to provide that $\mathfrak{S}_{\mathcal{V}}(N) > 0$. In this case we write

$$\mathcal{W} =: \{p^{e_p} \mid p \in \mathcal{P}\}\,, \tag{3}$$

instead of $\mathcal{V}$, where $\mathcal{P}$ is a subset of the prime numbers and (e_p) is a sequence of positive integers. Recall that we assume that there is some $\delta > 0$ such that

$$\sum_{p \in \mathcal{P}} \frac{1}{(p^{e_p})^{1-\delta}} < \infty. \tag{4}$$

Without loss of generality, we can restrict ourselves to the case that $e_p \in \{1, 2\}$ for all $p^{e_p} \in \mathcal{W}$, since a solution with the variables not divisible by p^2 implies a solution with the variables not divisible by p^r with $r \geq 2$.

Next, we define a condition that is necessary for $\mathfrak{S}_{\mathcal{W}}(N) > 0$ to hold. We define τ by $p^\tau || k$ and let $\sigma := \tau + 1$ if $p \neq 2$ and $\sigma := \tau + 2$ if $p = 2$. For an integer N, we say that $(\mathcal{W}, N)$ satisfies Condition C if the following two conditions hold:

- For each prime p such that $p \in \mathcal{W}$ (i.e. $e_p = 1$) with $p \leq (k-1)^4 + 8k$ there is a solution of
$$x_1^k + \ldots + x_s^k \equiv N \bmod p^\sigma$$
with $p \nmid x_1 \cdots x_s$.
- If $k = 2$ and $4 \in \mathcal{W}$, there is a solution of
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv N - 1 \bmod 8$$
such that $4 \nmid x_i$ for $i = 1, 2, 3, 4$.

Theorem 2. *Let a set of prime powers $\mathcal{W}$ be defined by* (3)*. If there is some $\delta > 0$ such that* (4) *and Condition C hold, we have*

$$\mathfrak{S}_{\mathcal{W}}(N) > 0.$$

If we take $\mathcal{W}$ as the set of the squares of all primes, we obtain the following corollary.

Corollary 1. *Let $s > 2^k$. Then every sufficiently large integer can be represented as a sum of s k–th powers of squarefree integers if $k \geq 3$. If $k = 2$, the same holds if there is a solution of $x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv N - 1 \mod 8$ with $4 \nmid x_i$ for $i = 1, 2, 3, 4$.*

If we sieve with primes, we have to be aware that the first part of Condition C does not hold in general. For example, if $k - 1 = p \in \mathcal{W}$, then $x^k \equiv 1 \mod p$ for all integers x with $p \nmid x$. Thus $N \equiv s \mod p$ is necessary for Condition C. If $2k + 1 = p \in \mathcal{W}$, then $x^k \equiv \pm 1 \mod p$ for $p \nmid x$ and Condition C cannot hold if s is even and $N \equiv 0 \mod p$. We illustrate this for the case $k = 2$. In this case, we have to verify Condition C for $p \leq 17$ by an easy computation.

Corollary 2. *Let $\mathcal{P}$ be a set of prime numbers and assume that there is some $\delta > 0$ such that*

$$\sum_{p\in\mathcal{P}} \frac{1}{p^{1-\delta}}$$

converges. Let $N \in \mathbb{N}$ and assume $N \equiv 5 \mod 8$ if $2 \in \mathcal{P}$ and $N \equiv 2 \mod 3$ if $3 \in \mathcal{P}$. If N is sufficiently large, it can be represented as a sum of five squares of integers not being divisible by any prime of $\mathcal{P}$.

To prove Theorem 1, we make use of the classical circle method. The main part of this paper is the proof of Theorem 2 which is motivated by the work of Baker and Brüdern [2].

2. Preliminary considerations

Let $P := N^{1/k}$ and let $\mathcal{V}$ be a set of pairwise coprime positive integers not containing 1. We define

$$f(\alpha) = \sum_{\substack{x\le P\\ \chi_{\mathcal{V}}(x)=1}} e(\alpha x^k). \tag{5}$$

As usual, $e(\theta)$ stands for $e^{2\pi i\theta}$. Recall that $R_{\mathcal{V}}(N)$ is the number of solutions of (2) with $\chi_{\mathcal{V}}(x_1) = \ldots = \chi_{\mathcal{V}}(x_s) = 1$. Thus

$$R_{\mathcal{V}}(N) = \int_0^1 f(\alpha)^s e(-\alpha N)d\alpha. \tag{6}$$

We define

$$\Pi(\mathcal{V}) := \left\{ n = \prod_{v\in\mathcal{V}'} v \,:\, \mathcal{V}' \text{ is a finite subset of } \mathcal{V} \right\}$$

and

$$\mu_{\mathcal{V}}(n) := \begin{cases} (-1)^{\#\mathcal{V}'}, & \text{if } n \in \Pi(\mathcal{V}) \text{ with } n = \prod_{v\in\mathcal{V}'} v,\\ 0\,, & \text{otherwise,}\end{cases}$$

a variant of the well known Möbius μ-function. Notice that $1 \in \Pi(\mathcal{V})$, since we define the empty product as 1. Similarly to $\mu^2(n) = \sum_{d^2|n} \mu(n)$, the convolution formula

$$\chi_{\mathcal{V}}(n) = \sum_{\substack{m\in\Pi(\mathcal{V})\\ m|n}} \mu_{\mathcal{V}}(m)$$

holds and is easy to verify. Thus

$$f(\alpha) = \sum_{\substack{d \le P \\ d \in \Pi(\mathcal{V})}} \mu_{\mathcal{V}}(d) \sum_{x \le P/d} e\left(\alpha d^k x^k\right). \tag{7}$$

Lemma 1. *Let $0 \le \delta < 1$. If $\sum_{v \in \mathcal{V}} 1/v^{1-\delta}$ is finite, then*

$$\sum_{\substack{d \ge 1 \\ d\xi \in \Pi(\mathcal{V})}} \frac{1}{d^{1-\delta}} \ll 1$$

holds uniformly for all $\xi \in \mathbb{N}$.

Proof. For arbitrary $\xi \in \mathbb{N}$ we define

$$\xi_0 := \prod_{\substack{v \in \mathcal{V} \\ (v,\xi) > 1}} \frac{v}{(v,\xi)}$$

and

$$\mathcal{V}_\xi := \{v \in \mathcal{V} | (v,\xi) = 1\} \cup \{\xi_0\}.$$

Let $d \ge 1$. If $d\xi \in \Pi(\mathcal{V})$, there is a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that

$$d\xi = \prod_{\substack{v \in \mathcal{V}' \\ (v,\xi) = 1}} v \prod_{\substack{v \in \mathcal{V}' \\ (v,\xi) > 1}} v. \tag{8}$$

Since $\mathcal{V}$ is a set of pairwise coprime integers, one has

$$\prod_{\substack{v \in \mathcal{V}' \\ (v,\xi) > 1}} v = \prod_{\substack{v \in \mathcal{V} \\ (v,\xi) > 1}} v = \xi \prod_{\substack{v \in \mathcal{V} \\ (v,\xi) > 1}} \frac{v}{(v,\xi)} = \xi\,\xi_0.$$

Hence we get by (8) that

$$d = \xi_0 \prod_{\substack{v \in \mathcal{V}' \\ (v,\xi) = 1}} v.$$

Thus $d\xi \in \Pi(\mathcal{V})$ implies $d \in \Pi(\mathcal{V}_\xi)$ and consequently

$$\sum_{\substack{d \ge 1 \\ d\xi \in \Pi(\mathcal{V})}} \frac{1}{d^{1-\delta}} \le \sum_{\substack{d \ge 1 \\ d \in \Pi(\mathcal{V}_\xi)}} \frac{1}{d^{1-\delta}}. \tag{9}$$

One has

$$\sum_{v \in \mathcal{V}_\xi} \log\left(1 + \frac{1}{v^{1-\delta}}\right) \le \sum_{v \in \mathcal{V}_\xi} \frac{1}{v^{1-\delta}} \le \sum_{v \in \mathcal{V}} \frac{1}{v^{1-\delta}} + 1. \tag{10}$$

By our assumption, the right hand side of (10) is finite. Thus

$$\exp\left(\sum_{v\in\mathcal{V}_\xi}\log\left(1+\frac{1}{v^{1-\delta}}\right)\right)=\prod_{v\in\mathcal{V}_\xi}\left(1+\frac{1}{v^{1-\delta}}\right)=\sum_{d\in\Pi(\mathcal{V}_\xi)}\frac{1}{d^{1-\delta}}$$

is finite and the lemma follows from (9). □

Besides, if $\sum_{v\in\mathcal{V}}1/v^{1-\delta}$ is finite, then

$$\sum_{\substack{d\le Y\\ d\in\Pi(\mathcal{V})}}1\ll Y^{1-\delta},\qquad\sum_{\substack{d>Y\\ d\in\Pi(\mathcal{V})}}\frac{1}{d}\ll Y^{-\delta},\tag{11}$$

and

$$\sum_{d\in\Pi(\mathcal{V})}\frac{\mu_{\mathcal{V}}(d)}{d}=\prod_{v\in\mathcal{V}}\left(1-\frac{1}{v}\right)\tag{12}$$

holds.

3. Asymptotic formula

Let $\eta>0$ be sufficiently small and be specified later. As in the classical case of Waring's Problem (see e.g., [5, page 15 ff.]), let the major arc $\mathfrak{M}$ be the union of the major arcs

$$\mathfrak{M}(q,a):=\left\{\alpha\in[0,1]\,:\,\left|\alpha-\frac{a}{q}\right|\le P^{-k+\eta}\right\}$$

with $1\le a\le q\le P^\eta$ and $(a,q)=1$, and let $\mathfrak{m}:=[0,1]\backslash\mathfrak{M}$ be the minor arcs.

3.1. Minor arcs

Our next aim is to show that the integral in (6) restricted to the minor arcs $\mathfrak{m}$ is $O(N^{s/k-1-\rho})$. To do so, we need a variant of Weyl's inequality.

For coprime integers a,q let $\alpha\in\mathbb{R}$ satisfy $|\alpha-a/q|\le q^{-2}$. Let $K=2^{k-1}$. By Weyl's classical inequality (see e.g., [13, p. 12]), we get

$$\begin{aligned}&\left|\sum_{x\le P/d}e\left(\alpha d^kx^k\right)\right|^K\\ \ll\;&(P/d)^{K-k+\varepsilon}\left((P/d)^{k-1}+\sum_{h\le k!(P/d)^{k-1}}\min\left\{(P/d)^k\,h^{-1},\|\alpha d^kh\|^{-1}\right\}\right).\end{aligned}$$

We substitute $y := d^k h$ in the h-sum above and extend the summation. Thus, the h-sum is less than

$$\sum_{y \le k! P^k} \min\left\{P^k y^{-1}, \|\alpha y\|^{-1}\right\} \ll P^{k+\varepsilon}\left(\frac{1}{q} + \frac{q}{P^k}\right),$$

where we made use of [13, Lemma 2.2]. Thus we showed

$$\sum_{x \le P/d} e\left(\alpha d^k x^k\right) \ll P^{1+\varepsilon}\left(\frac{1}{q} + \frac{q}{P^k}\right)^{1/K}. \tag{13}$$

Notice that the implicit O–constant does not depend on d.

We assume that $s > 2^k$. We have

$$\int_{\mathfrak{m}} f(\alpha)^s e(-\alpha N) d\alpha \le \left(\sup_{\alpha \in \mathfrak{m}} \{|f(\alpha)|\}\right)^{s-2^k} \int_0^1 |f(\alpha)|^{2^k} d\alpha. \tag{14}$$

Recall formula (7). Let $\alpha \in \mathfrak{m}$. By Dirichlet's Approximation Theorem, there exist coprime integers a, q with $q \le P^{k-\eta}$ and $|\alpha - a/q| \le 1/qP^{k-\eta}$. Since $q \le P^\eta$ implies $\alpha \in \mathfrak{M}(a,q)$, we have $P^\eta < q \le P^{k-\eta}$. Let $\beta > 0$ such that $\beta\delta < \eta/K$. By (13), we have

$$\sum_{\substack{d \le P^\beta \\ d \in \Pi(\mathcal{V})}} \sum_{x \le P/d} e\left(\alpha d^k x^k\right) \ll P^{1-\eta/K+\beta\delta+\varepsilon}. \tag{15}$$

On the other hand,

$$\sum_{\substack{d > P^\beta \\ d \in \Pi(\mathcal{V})}} \sum_{x \le P/d} e\left(\alpha d^k x^k\right) \ll P^{1-\delta\beta} \tag{16}$$

by trivial estimates and (11). Recall $\beta\delta < \eta/K$. By this (15), (16) we can bound the supremum in (14). By Hua's Lemma, the integral in (14) is $O\left(P^{2^k-k+\varepsilon}\right)$. Thus we obtain the following lemma.

Lemma 2. *Suppose that $s > 2^k$. Then there is some $\rho > 0$ only depending on η, s and k such that*

$$\int_{\mathfrak{m}} f(\alpha)^s e(-\alpha N) d\alpha \ll N^{s/k-1-\rho}.$$

3.2. Major arcs

We define

$$S(q,b) := \sum_{m=1}^{q} e\left(\frac{bm^k}{q}\right),$$

and the singualr series

$$\mathfrak{S}_{\mathcal{V}}(N) := \sum_{q\geq 1} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\sum_{d\in\Pi(\mathcal{V})} \mu_{\mathcal{V}}(d) \frac{S(q,ad^k)}{dq}\right)^s e\left(-\frac{a}{q}N\right).$$

Lemma 3. *The singular series $\mathfrak{S}_{\mathcal{V}}(N)$ is absolutely convergent.*

P r o o f. We go along the lines of [2, p. 5]. First, we need a bound for $|S(q,b)|$, where b, q are not necessarily coprime. One has

$$S(q,b) = (a,q) S\left(q/(a,q), b/(a,q)\right)$$

and by [5, Lemma 6.4] we get

$$S(q,b) \ll q^{1-1/k}(b,q)^{1/k}. \tag{17}$$

Let

$$\mathcal{S}_{q,a} := \sum_{d\in\Pi(\mathcal{V})} \mu_{\mathcal{V}}(n) \frac{S(q,ad^k)}{qd}.$$

One has

$$\mathcal{S}_{q,a} \leq \sum_{d\in\Pi(\mathcal{V})} \frac{|S(q,ad^k)|}{qd} = \sum_{\xi|q} \sum_{\substack{d\in\Pi(\mathcal{V})\\(d,q)=\xi}} \frac{|S(q,ad^k)|}{qd} = \sum_{\xi|q} \sum_{\substack{\hat{d}\geq 1\\ \hat{d}\xi\in\Pi(\mathcal{V})\\(\hat{d},q/\xi)=1}} \frac{|S(q,a\hat{d}^k\xi^k)|}{q\hat{d}\xi}.$$

It is easy to see that $S(q,a\hat{d}^k\xi^k) = S(q,a\xi^k)$. Hence

$$\mathcal{S}_{q,a} \leq \sum_{\xi|q} \frac{|S(q,a\xi^k)|}{\xi q} \sum_{\substack{\hat{d}\geq 1\\ \hat{d}\xi\in\Pi(\mathcal{V})\\(\hat{d},q/\xi)=1}} \frac{1}{\hat{d}} \ll q^{-1/k+\varepsilon} \tag{18}$$

by (17) and Lemma 1. Now, the lemma follows. In particular we have

$$\sum_{q\geq Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\sum_{d\in\Pi(\mathcal{V})} \mu_{\mathcal{V}}(d) \frac{S(q,ad^k)}{dq}\right)^s e\left(-\frac{a}{q}N\right) \ll Q^{2-s/k+\varepsilon} \tag{19}$$

for any $Q > 0$, which will be useful later. □

Now, we want to prove Theorem 1. By Lemma 2, it remains to study the the mayor arcs $\mathfrak{M}$. The integral in (6) restricted to $\mathfrak{M}$ equals

$$\sum_{\substack{d_1,\dots d_s \le P \\ d_1,\dots d_s \in \Pi(\mathcal{V})}} \mu_{\mathcal{V}}(d_1)\cdots\mu_{\mathcal{V}}(d_s) \int_{\mathfrak{M}} f_{d_1}(\alpha)\cdots f_{d_s}(\alpha) e(-\alpha N)\, d\alpha \tag{20}$$

with

$$f_d(\alpha) := \sum_{x \le P/d} e(\alpha d^k x^k).$$

Let $\gamma < 0$. It is an easy exercise to show that we can restrict the summation over $d_1, \dots d_s \le P$ in the last displayed formula to $d_1, \dots d_s \le P^\gamma$ at the cost of an error term $O\left(N^{s/k-1-\rho'}\right)$ for some $\rho' > 0$.

Let $1 \le a \le q \le P^\eta$ with $(a,q) = 1$ and $\alpha \in \mathfrak{M}(a,q)$. Define $\beta := \alpha - a/q$,

$$S(q,b) := \sum_{m=1}^{q} e\left(\frac{bm^k}{q}\right),$$

and

$$I(\beta) := \int_0^P e\left(\beta u^k\right)\, du.$$

The formula

$$\sum_{d \le P} e\left(\alpha x^k\right) = \frac{S(q,a)}{q} I(\beta) + O\left(P^{2\eta}\right)$$

is well known (see e.g [5, Lemma 4.2]). It is straightforward to generalize this result and show

$$f_d(\alpha) = \frac{S(q,ad^k)}{qd} I(\beta) + O\left(P^{2\eta}\right) \tag{21}$$

uniformly in all integers d. Note that the error term does not depend on d.

By (21), a perusal of the proof of [5, Lemma 4.3] yields

$$\begin{aligned} \int_{\mathfrak{M}} f_{d_1}(\alpha)\cdots f_{d_s}(\alpha) e(-\alpha N)\, d\alpha = \quad & \mathfrak{S}_{d_1,\dots,d_s}\left(P^\eta, N\right) J\left(P^\eta\right) N^{s/k-1} \\ & + O\left(P^{s-k-(1-5\eta)}\right) \end{aligned} \tag{22}$$

uniformly for all integers $d_1, \dots, d_s$ with

$$\mathfrak{S}_{d_1,\dots,d_s}(P^\eta, N) := \sum_{q \le P^\eta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S(q, ad_1^k)}{qd_1} \cdots \frac{S(q, ad_s^k)}{qd_s} e\left(-\frac{a}{q}N\right),$$

and

$$J(P^\eta) := \int_{|\beta| < P^\eta} \left(\int_0^1 e(\beta u^k)\, du \right)^s e(-\beta)\, d\beta. \tag{23}$$

We multiply both sides of (22) by $\mu_{\mathcal{V}}(d_1) \cdots \mu_{\mathcal{V}}(d_s)$ and sum over $d_1, \dots, d_s \le P^\gamma, d_1, \dots, d_s \in \Pi(\mathcal{V})$, and obtain that $R(N)$ equals

$$\begin{aligned} J(P^\eta) \sum_{q \le P^\eta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\sum_{\substack{d \le P^\gamma \\ d \in \Pi(\mathcal{V})}} \mu_{\mathcal{V}}(d) \frac{S(q, ad^k)}{qd} \right)^s e\left(-\frac{a}{q}N\right) N^{s/k-1} \\ + O\left(P^{s-k-(1-5\eta-s\gamma)} + P^{s-k-\rho'}\right). \end{aligned} \tag{24}$$

Since

$$\sum_{\substack{d > N^\gamma \\ d \in \Pi(\mathcal{V})}} \frac{|S(a, qd^k)|}{qd} \ll P^{-\gamma\delta},$$

the completion of the d–sum in (24) engenders an additional error term

$$O\left(P^{s-k-(\gamma\delta-2\eta)}\right).$$

Hence, we have to force that $2\eta < \gamma\delta$ holds. We choose η and γ small enough so that this inequality and $5\eta + s\gamma < 1$ are both fulfilled. By (19), the completion of the singular series, e.g., the completion of the q–sum in (24), creates an additional error term which is negligible. Finally, by [5, p. 21], we can replace $J(P^\eta)$ by $J := J(\infty) < \infty$ at the cost of an negligible error term and Theorem 1 follows.

4. The singular series

In this section, we restrict $\mathcal{V}$ to be a set of prime powers and use the symbol $\mathcal{W}$ instead.

We consider a more general singular series and generalize the procedure presented in [2] and [12]. We go along the lines of [2, Section 5]. For shortness, bold

symbols are reserved to denote an element in $\mathbb{N}^s$, e.g. we write $\mathbf{n} = (n_1, \dots, n_s)$ for an s-tuple of integers. Let $\varphi \in \mathbb{Z}[\mathbf{x}]$. We write

$$\begin{aligned} \varphi_{\mathbf{h}}(\mathbf{x}) &:= \varphi(h_1 x_1, \dots, h_s x_s), \\ \mathbf{h}\,\mathbf{x} &:= (h_1 x_1, \dots, h_s x_s), \end{aligned}$$

and further

$$\mu_{\mathcal{W}}(\mathbf{h}) := \mu_{\mathcal{W}}(h_1) \cdots \mu_{\mathcal{W}}(h_s)$$

for $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{h} = (h_1, \dots, h_s)$. For the sake of a short notation, if we use the index j in an expression, it is understood that j ranges over $1, \dots, s$. For instance $x_j \in A$ means $x_1 \in A, \dots, x_s \in A$ for $A \subset \mathbb{N}$. The symbol j is reserved for this usage only.

Besides, introduce

$$T(\varphi, q) := \sum_{x_j=1}^{q} e\left(\frac{\varphi(\mathbf{x})}{q}\right),$$

and

$$A(\mathbf{d}, q) := \sum_{\substack{a=1 \\ (a,q)=1}}^{q} T(a\varphi_{\mathbf{d}}, q).$$

We fix $\varphi \in \mathbb{Z}[\mathbf{x}]$ to be the polynomial

$$\varphi(x_1, \dots, x_s) := x_1^k + \ldots + x_s^k - N.$$

With this definitions, it is easy to see that we have

$$\mathfrak{S}_P(N) = \sum_{q \geq 1} \frac{1}{q^s} \sum_{d_j \in \Pi(\mathcal{W})} \frac{\mu_{\mathcal{W}}(\mathbf{d})}{d_1 \cdots d_s} A(\mathbf{d}, q), \tag{25}$$

where the singular series was defined before Lemma 3.

4.1. Factorization

For every $q \in \mathbb{N}$, we subdivide $\mathcal{W}$ into two disjoint sets

$$\mathcal{W}_q^+ := \{\, v \,|\, v \in \mathcal{W} \text{ and } (v, q) > 1 \,\}$$

and

$$\mathcal{W}_q^- := \{\, v \,|\, v \in \mathcal{W} \text{ and } (v, q) = 1 \,\}.$$

We omit the proof of the following lemma, which is trivial.

Lemma 4. *Let $q, q_1, q_2 \in \mathbb{N}$ with $(q_1, q_2) = 1$.*

(1): *Every integer $g \in \Pi(\mathcal{W})$ can be factorized uniquely in the form $g = dt$ with $d \in \Pi\left(\mathcal{W}_q^+\right)$ and $t \in \Pi\left(\mathcal{W}_q^-\right)$. In particular, we have $(d, t) = 1$ and $(t, q) = 1$.*

(2): *Every integer $g \in \Pi\left(\mathcal{W}_{q_1 q_2}^+\right)$ can be factorized uniquely in the form $g = dt$ with $d \in \Pi\left(\mathcal{W}_{q_1}^+\right)$ and $t \in \Pi\left(\mathcal{W}_{q_2}^+\right)$. One further has $(d, t) = 1$, $(d, q_2) = 1$ and $(t, q_1) = 1$.*

Remark 1. Note that the second part of this lemma is not valid if $\mathcal{W}$ is replaced by a set $\mathcal{V}$ of arbitrary coprime integers. For instance, let $\mathcal{V} = \{6\}$, then $6 \in \Pi\left(\mathcal{V}_6^+\right)$ does not factorize over $\Pi\left(\mathcal{V}_2^+\right) \cup \Pi\left(\mathcal{V}_3^+\right)$. This is the reason why we restrict ourselves to $\mathcal{W}$.

Lemma 5. *Let $q \in \mathbb{N}$. One has*

$$\sum_{d_j \in \Pi(\mathcal{W})} \frac{\mu_{\mathcal{W}}(\mathbf{d})}{d_1 \cdots d_s} A(\mathbf{d}, q) = C \prod_{v \in \mathcal{W}_q^+} \left(1 - \frac{1}{v}\right)^{-s} B(q), \tag{26}$$

where

$$B(q) := \sum_{d_j \in \Pi\left(\mathcal{W}_q^+\right)} \mu_{\mathcal{W}}(\mathbf{d}) \frac{A(\mathbf{d}, q)}{d_1 \cdots d_s},$$

and $C > 0$ is a constant depending only on the set $\mathcal{W}$. The function $B(q)$ is multiplicative.

Proof. This lemma is a generalization of [2, Lemma 7]. By the first part of Lemma 4, the left hand side of (26) equals

$$\sum_{d_j \in \Pi\left(\mathcal{W}_q^+\right)} \sum_{t_j \in \Pi\left(\mathcal{W}_q^-\right)} \frac{\mu_{\mathcal{W}}(\mathbf{dt})}{d_1 \cdots d_s \, t_1 \cdots t_s} A(\mathbf{dt}, q). \tag{27}$$

Since $(d_j, t_j) = 1$, we have $\mu_{\mathcal{W}}(\mathbf{dt}) = \mu_{\mathcal{W}}(\mathbf{d}) \mu_{\mathcal{W}}(\mathbf{t})$ and $A(\mathbf{dt}, q) = A(\mathbf{d}, q)$ by the definition of $A(\mathbf{d}, q)$ and since $(t_j, q) = 1$ by the first part of Lemma 4. Hence (27) simplifies to

$$\sum_{d_j \in \Pi\left(\mathcal{W}_q^+\right)} \mu_{\mathcal{W}}(\mathbf{d}) \frac{A(\mathbf{d}, q)}{d_1 \cdots d_s} \sum_{t_j \in \Pi\left(\mathcal{W}_q^-\right)} \frac{\mu_{\mathcal{W}}(\mathbf{t})}{t_1 \cdots t_s}.$$

The inner multiple sum is the s–th power of

$$\sum_{t \in \Pi\left(\mathcal{W}_q^-\right)} \frac{\mu_{\mathcal{W}}(t)}{t} = \prod_{\substack{v \in \mathcal{W} \\ (v,q)=1}} \left(1 - \frac{1}{v}\right) = C^{1/s} \prod_{\substack{v \in \mathcal{W} \\ (v,q)>1}} \left(1 - \frac{1}{v}\right)^{-1}$$

with

$$C^{1/s} = \prod_{v \in \mathcal{W}} \left(1 - \frac{1}{v}\right) > 0.$$

The multiplicativity of $B(q)$ is a simple generalization of [2, Lemma 9]. □

Recall that by Lemma 3 the singular series $\mathfrak{S}_{\mathcal{W}}(N)$ is absolutely convergent. By (25) we get

$$\mathfrak{S}_{\mathcal{W}}(N) = C \sum_{q \geq 1} D(q)$$

with

$$D(q) = \frac{1}{q^s} \prod_{v \in \mathcal{W}_q^+} \left(1 - \frac{1}{v}\right)^{-s} B(q).$$

On the account of the second part of Lemma 4 and the multiplicativity of $B(q)$, the function $D(q)$ is multiplicative. Thus

$$\mathfrak{S}_{\mathcal{W}}(N) = C \prod_{p \text{ prime}} \left(\sum_{n \geq 0} D(p^n) \right). \tag{28}$$

We have to distinguish two cases in order to determine the value of the factor $\sum_{n \geq 0} D(p^n)$. If $p \nmid v$ for all $v \in \mathcal{W}$, then $\mathcal{P}_{p^n}^+ = \emptyset$ and $\Pi(\mathcal{P}_{p^n}^+) = \{1\}$. In this case $\sum_{n \geq 0} D(p^n)$ equals the factor usually denoted by

$$\chi(p) = \sum_{n \geq 0} \sum_{\substack{1 \leq a < p^n \\ p \nmid a}} \left(\frac{S(p^n, a)}{p^n}\right)^s e\left(-\frac{a}{p^n} N\right)$$

associated with the singular series $\mathfrak{S} = \prod_p \chi(p)$ of the classical Waring's Problem. Thus we

$$\mathfrak{S}_{\mathcal{W}}(N) = C \prod_{\substack{p \text{ prime} \\ p \nmid v \, \forall v \in \mathcal{W}}} \chi(p) \prod_{\substack{p \text{ prime} \\ p \sim \mathcal{W}}} \kappa(p),$$

where

$$\kappa(p) := \sum_{n \geq 0} D(p^n)$$

and the abbreviation $p \sim \mathcal{W}$ is equal to $\exists v \in \mathcal{W} : p | v$. In this case let v_p denote the unique element of $\mathcal{W}$ such that $p | v_p$. Notice that that $0 < \prod_p \chi(p) \ll 1$ (see

e.g., [5, Lemma 5.2 and Lemma 5.6]). Let $p \sim \mathcal{W}$. We have

$$\kappa(p) = 1 + \left(1 - \frac{1}{v_p}\right)^s \sum_{n \geq 1} p^{-sn} B(p^n).$$

Let $n \geq 1$. By the definition of $B(p^n)$ and (18) we have

$$p^{-sn} B(p^n) \ll p^{-n(s/k-1-\varepsilon)}.$$

Hence

$$\kappa(p) = 1 + O\left(p^{-s/k+1+\varepsilon}\right),$$

and consequently (see e.g., [5, Corollary to Lemma 5.2]) there is a p_0 depending only on s, k such that

$$\frac{1}{2} \leq \prod_{p > p_0} \kappa(p) \leq \frac{3}{2}.$$

Therefore, we have proved the following lemma.

Lemma 6. *There is some $p_0 \in \mathbb{N}$, depending only on s, k, such that if*

$$\kappa(p) > 0$$

for all $p \sim \mathcal{W}$ with $p \leq p_0$, then

$$\mathfrak{S}_{\mathcal{W}}(N) > 0.$$

4.2. Properties of $\kappa(p)$

For the remainder of the paper, we assume $p \sim \mathcal{W}$. We have $D(1) = 1$ and

$$D(p^n) = \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n} \Upsilon^s(a, p^n) e\left(-\frac{a}{p^n} N\right)$$

for $n \geq 1$ where

$$\Upsilon(a, p^n) := p^{-n} \left(1 - \frac{1}{v_p}\right)^{-1} \sum_{d \in \{1, v_p\}} \frac{\mu_{\mathcal{W}}(d)}{d} \sum_{x=1}^{p^n} e\left(\frac{a d^k x^k}{p^n}\right).$$

Lemma 7. *Let $p \sim \mathcal{W}$. Suppose $v_p = p^r$. One has*

$$\kappa(p) = \left(1 - \frac{1}{v_p}\right)^{-s} \lim_{l \to \infty} \frac{M(p^\ell)}{p^{\ell(s-1)}},$$

where $\ell = 2r(k-1) - 1$ and $M(p^\ell)$ is defined as the number of solutions of

$$x_1^k + \ldots + x_s^k \equiv N \bmod p^\ell \tag{29}$$

with $v_p \nmid x_j$ and $0 < x_j < p^\ell$.

P r o o f. The proof goes along the lines of [2, Lemma 10]. We give, however, a detailed proof here since there are some minor errors in the proof in [2]. Let $\ell \geq r$. For $\mathbf{d} \in \mathbb{N}^s$ let $M(\mathbf{d}, p^\ell)$ denote the number of solutions modulo p^ℓ of (29) with $d_j | x_j$. Recall that $\mu_{\mathcal{W}}(1) = 1$ and $\mu_{\mathcal{W}}(v) = -1$ for all $v \in \mathcal{W}$. Thus

$$M(p^\ell) = \sum_{d_j \in \{1, v_p\}} \mu_{\mathcal{W}}(\mathbf{d}) M(\mathbf{d}, p^\ell) \tag{30}$$

by the inclusion-exclusion principle. For $d_j \in \{1, v_p\}$, let

$$L(\mathbf{d}, p^\ell) := \sum_{y=1}^{p^\ell} \sum_{x_j=1}^{p^\ell} e\left(\frac{y}{p^\ell} \varphi_{\mathbf{d}}(\mathbf{x})\right).$$

On the one hand,

$$\begin{aligned} L(\mathbf{d}, p^\ell) &= d_1 \cdots d_s \sum_{y=1}^{p^\ell} \sum_{z_j=1}^{p^\ell / d_j} e\left(\frac{y}{p^\ell} \left((d_1 z_1)^k + \ldots + (d_s z_s)^k - N\right)\right) \\ &= d_1 \cdots d_s \, p^\ell \, M(\mathbf{d}, p^\ell); \end{aligned} \tag{31}$$

on the other hand,

$$\begin{aligned} L(\mathbf{d}, p^\ell) &= \sum_{n=0}^{\ell} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n} \sum_{x_j=1}^{p^\ell} e\left(\frac{a p^{\ell-n}}{p^\ell} \varphi_{\mathbf{d}}(\mathbf{x})\right) \\ &= \sum_{n=0}^{\ell} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n} \left(p^{\ell-n}\right)^s \sum_{x_j=1}^{p^n} e\left(\frac{a}{p^n} \varphi_{\mathbf{d}}(\mathbf{x})\right). \end{aligned} \tag{32}$$

Combining (31), (32) and (30), we deduce

$$M(p^\ell) = p^{\ell(s-1)} \sum_{n=0}^{\ell} p^{-ns} \sum_{d_j \in \{1, v_p\}} \frac{\mu_{\mathcal{W}}(\mathbf{d})}{d_1 \cdots d_s} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n} \sum_{x_j=1}^{p^n} e\left(\frac{a}{p^n} \varphi_{\mathbf{d}}(\mathbf{x})\right). \tag{33}$$

Recall that $\Pi\left(\mathcal{P}^+_{p^n}\right) = \{1, v_p\}$. If $n \geq 1$, we have

$$\begin{aligned} &p^{-ns} \sum_{d_j \in \{1, v_p\}} \frac{\mu_{\mathcal{W}}(\mathbf{d})}{d_1 \cdots d_s} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n} \sum_{x_j=1}^{p^n} e\left(\frac{a}{p^n} \varphi_{\mathbf{d}}(\mathbf{x})\right) \\ &= \left(1 - \frac{1}{v_p}\right)^s \Upsilon^s(a, p^n) e\left(-\frac{a}{p^n} N\right). \end{aligned} \tag{34}$$

The summand corresponding to $n = 0$ in the n–sum of (33) equals

$$\sum_{d_j \in \{1, v_p\}} \frac{\mu_{\mathcal{W}}(\mathbf{d})}{d_1 \cdots d_s} = \sum_{y=1}^{s} \left(-p^{-1}\right)^y \binom{s}{y} = \left(1 - \frac{1}{v_p}\right)^s.$$

By this, togheter with (34) and (33), we get

$$\begin{aligned} M(p^\ell) &= p^{\ell(s-1)} \left(1 - \frac{1}{v_p}\right)^s \left(1 + \sum_{n=1}^{\ell} \Upsilon^s(a, p^n) e\left(-\frac{a}{p^n} N\right)\right) \\ &= p^{\ell(s-1)} \left(1 - \frac{1}{v_p}\right)^s \sum_{n=0}^{\ell} D(p^n) \end{aligned}$$

and the lemma follows. □

4.3. Proof of Theorem 2

Lemma 8. *Let $p \sim \mathcal{W}$ with $p^2 = v_p$.*

- *If $p \nmid k$, one has $\kappa(p) > 0$.*
- *If $p|k$, define τ by $p^\tau || k$ and let $\sigma = \tau + 1$ if $p \neq 2$ and $\sigma = \tau + 2$ if $p = 2$. Then $\kappa(p) > 0$ whenever*

$$x_1^k + \ldots + x_s^k \equiv N \bmod p^\sigma$$

has a solution with $p \nmid x_1$ and $p^2 \nmid x_i$ for $i = 2, \ldots, s$.

Proof. By Lemma 7, it is sufficient to show that $M(p^\ell) > C_p\, p^{\ell(s-1)}$ for $\ell > 2 + \sigma$ where $C_p > 0$ is some constant denpending only on p.

Let $p \nmid k$. It is known (see e.g. [13, Lemma 2.15]) that for every integer N there are integers $1 \leq x_j \leq p$ with $p \nmid x_1$ such that

$$x_1^k + \ldots + x_s^k \equiv N \bmod p.$$

Clearly, $p^2 \nmid x_i$ for $i = 2, \ldots, s$. By [5, Lemma 5.4] there is an integer x such that

$$x^k + x_2^k + \ldots + x_s^k \equiv N \bmod p^2,$$

where $x \equiv x_1\,(p)$ and hence $p \nmid x$. As in [2, page 20], we can obtain by Hensel's Lemma (see e.g. [11, page 87]) the existence of $p^{(s-1)(\ell-2)}$ solutions of

$$y_1^k + \ldots + y_s^k \equiv N \bmod p^\ell \tag{35}$$

with $p \nmid y_1$, $p^2 \nmid y_i$ for $i = 2, \ldots, s$ and the lemma follows in the case $p \nmid k$.

If $p|k$, we assume a solution of $x_1^k + \ldots x_s^k \equiv N \mod p^\sigma$ with $p \nmid x_1$ and $p^2 \nmid x_i$ for $i = 2, \ldots, s$. We can lift this solution as in [5, Lemma 5.5] and obtain $p^{(s-1)(\ell-\sigma)}$ solutions of (35). □

The following lemma is useful to prove an analog to Lemma 8 when $p \in \mathcal{W}$, i.e., $v_p = p$.

Lemma 9. *Let $p, n \in \mathbb{N}$ with p prime and $p \nmid n$. Then*

$$x^k + y^k \equiv n \bmod p \tag{36}$$

has a solution with $p \nmid xy$ whenever $p > (k-1)^4 + 8k$.

Proof. Denote by ω the number of solutions modulo p of (36) having $p \nmid xy$, and let Ω stand for the number of solutions modulo p of (36) with x, y arbitrary.

By [10, Theorem 6.37], one has

$$\Omega \geq p - (k-1)^2\sqrt{p}.$$

Since the number of $(x, y) \in \{0, \ldots, p-1\}^2$ such that $x^k + y^k \equiv N\,(p)$ and $p|xy$ is at most $2k$, we have

$$\omega \geq p - (k-1)^2\sqrt{p} - 2k.$$

By a short computation we get that $\omega > 0$ if $p > (k-1)^4 + 8k$. □

Lemma 10. *Let $p \in \mathcal{W}$.*

- *If $p > (k-1)^4 + 8k$, then $\kappa(p) > 0$.*
- *If $p \leq (k-1)^4 + 8k$, then $\kappa(p) > 0$ whenever*

$$x_1^k + \ldots + x_s^k \equiv N \bmod p^\sigma$$

has a solution with $p \nmid x_1 \cdots x_s$, where σ is as in Lemma 8.

Proof. Notice that $p > (k-1)^4 + 8k$ implies $p \nmid k$. First we show that

$$x_1^k + \ldots + x_s^k \equiv N \bmod p \tag{37}$$

has a solution with $p \nmid x_1 \cdots x_s$ for all p with $p \nmid k$ and $p > (k-1)^4 + 8k$. For all $i \geq 4$ we take $x_i = 1$ and have to find a solution of

$$x_1^k + x_2^k \equiv N - (s-3) - x_3^k \bmod p,$$

such that $p \nmid x_1 x_2 x_3$. By Lemma 9, such a solution is guaranteed if there is some x_3 not divisible by p such that

$$N - (s-3) - x_3^k \not\equiv 0 \bmod p. \tag{38}$$

It is known that the number of k–th power residues modulo p is

$$\frac{p-1}{(k, p-1)} \geq \frac{(k-1)^4 + 8k}{k} \geq 3.$$

Thus there are integers $a \not\equiv b\,(p)$, both not divisible by p such that $a \equiv x^k \bmod p$ and $b \equiv x^k \bmod p$ have a solution. Thus (38) has a solution with $p \nmid x_3^k$.

It remains to show that a solution of

$$x_1^k + \ldots + x_s^k \equiv N \bmod p^\sigma$$

with $p \nmid x_1 \cdots x_s$ implies the existence of $p^{(s-1)(\ell-\sigma)}$ solutions of

$$y_1^k + \ldots + y_s^k \equiv N \bmod p^\ell$$

with $p \nmid y_1 \cdots y_s$ for all $\ell > \sigma$. Similarly to the proof of Lemma 8, this is a consequence of [5, Lemmas 5.4 and 5.5]. □

Now from Lemmas 6, 8 and 10, we deduce that $\mathfrak{S}_{\mathcal{W}}(N) > 0$ if the following conditions hold:

C_1: For all $p^2 \in \mathcal{W}$ with $p|k$ there is a solution of

$$x_1^k + \ldots + x_s^k \equiv N \bmod p^\sigma \tag{39}$$

with $p \nmid x_1$ and $p^2 \nmid x_i$ for $i = 2, \ldots, s$.

C_2: For all $p \in \mathcal{W}$ with $p \le (k-1)^4 + 8k$ there is a solution of (39) with $p \nmid x_1 \cdots x_s$.

Lemma 11. *If $p^2 \in \mathcal{W}$ with $p \nmid k$, then Condition C_1 holds unless $pk \neq 4$.*

Proof. Let $pk \neq 4$. It suffices to detect a solution of (39) with $p \nmid x_1$ and $p^2 \nmid x_i$ for $i = 2, \ldots, s$. Notice that $0 \equiv x^k \bmod p^\sigma$ has a solution $x = p$ if $\sigma \le k$. By the definition of σ, the condition $\sigma \le k$ holds unless $k = p = 2$.

Notice that $s > p^\sigma$ unless $k \le 4$. Thus if $k \ge 4$, we can construct a solution of (39) by taking $x_1 = 1$ and $x_i^k \in \{0, 1\}$, i.e., $x_i \in \{1, p\}$, suitable for $i = 1, \ldots s$.

It remains to investigate $p = k = 3$. A solution of (39) is found since $x^k \equiv 1, -1$ (9) for $3 \nmid x$ and $x^k \equiv 1, -1, 0$ (9) for $9 \nmid x$. □

Notice that the case $k = p = 2$ was investigated by Esterman [7]. Since $x_1^2 \equiv 1$ (8) if $2 \nmid x_1$, Condition C_1 is reduced to Condition A if $k = 2$ and $4 \in \mathcal{W}$. Now, Theorem 2 follows.

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Received November 26, 2009
Accepted January 14, 2010

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