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ON THE LATTICE DISCREPANCY OF ELLIPSOIDS OF ROTATION

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ABSTRACT. The objective of this paper is to prove that the number of lattice points $A_{\mathcal{E}_a}(t)$ in the ellipsoid

$$\frac{u_1^2+u_2^2}{a}+a^2\,u_3^2\le t^2$$

satisfies the asymptotic

$$A_{\mathcal{E}_a}(t)=\frac{4\pi}{3}t^3+O\left(t^{679/494+\varepsilon}\right),$$

for fixed a, large t, and any $\varepsilon>0$. This improves upon the error term $O\left(t^{11/8+\varepsilon}\right)$ known before.

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To Professor Ekkehard Krätzel on his 75th birthday

1. Introduction

Let $\mathcal{B}$ denote a compact convex body in three-dimensional Euclidean space $\mathbb{R}^3$ which contains the origin in its interior. Suppose that the boundary $\partial\mathcal{B}$ is smooth of class C^∞, with finite nonzero Gaussian curvature throughout. For t a large real parameter, we are interested in the number $A_{\mathcal{B}}(t)$ of integer points in a linearly dilated copy $t\mathcal{B}$, in particular in the size of the *lattice point discrepancy*

$$P_{\mathcal{B}}(t)=A_{\mathcal{B}}(t)-\mathrm{vol}(\mathcal{B})\,t^3. \tag{1.1}$$

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For enlightening expositions of the general *theory of lattice points in large domains*, see the monographs of M. Huxley [8], and of E. Krätzel [13], [14], as well as the survey article [11] by A. Ivić, M. Kühleitner, E. Krätzel, and the author.

Under the general conditions stated, a classic result of E. Hlawka [7] from 1950 tells us that

$$P_{\mathcal{B}}(t) = O\left(t^{3/2}\right), \tag{1.2}$$

as $t \to \infty$, the O-constant depending on $\mathcal{B}$. Some 40 years later, this was improved by E. Krätzel and the author [18], [19], to $O\left(t^{37/25}\right)$, resp.,

$$O\left(t^{25/17}(\log t)^{10/17}\right).$$

The sharpest result at the present state-of-art is due to W. Müller [24] and reads $P_{\mathcal{B}}(t) = O\left(t^{63/43+\varepsilon}\right)$. Note that $\frac{37}{25} = 1.48$, $\frac{25}{17} = 1.470\ldots$, and $\frac{63}{43} = 1.4651\ldots$

A much more precise result is known for the special case of the three-dimensional unit ball $\mathcal{B}_0$ ("sphere problem"): D.R. Heath-Brown [6] proved in 1999 that

$$P_{\mathcal{B}_0}(t) = O\left(t^{21/16+\varepsilon}\right), \qquad \left(\tfrac{21}{16} = 1.3125\right) \tag{1.3}$$

improving upon earlier works by I.M. Vinogradov [28] and F. Chamizo & H. Iwaniec [3], who had $O\left(t^{4/3}(\log t)^6\right)$, resp., $O\left(t^{29/22+\varepsilon}\right)$. Using very deep techniques from the theory of modular forms, F. Chamizo, E. Cristóbal and A. Ubis [4] recently generalized Heath-Brown's bound to arbitrary *rational* (origin-centered) ellipsoids.

On the other hand, F. Chamizo [2] investigated *bodies of rotation.* Adding to the assumptions stated the condition that $\mathcal{B}_{\mathrm{rot}}$ is invariant under rotations about one coordinate axis, he obtained

$$P_{\mathcal{B}_{\mathrm{rot}}}(t) = O\left(t^{11/8+\varepsilon}\right), \qquad \left(\tfrac{11}{8} = 1.375\right). \tag{1.4}$$

This was accomplished by an ingenious generalization of an approach originally developed by I.M. Vinogradov [27] to deal with the $\mathbb{R}^3$-sphere problem. Lower bounds for this lattice discrepancy have been given by M. Kühleitner [22], and by M. Kühleitner and the author [23]. E. Krätzel and the author [20] dealt with the special case of an *ellipsoid of rotation*

$$\mathcal{E}_a: \ \frac{u_1^2 + u_2^2}{a} + a^2 u_3^2 \le 1, \tag{1.5}$$

working out the dependance on the shape parameter a. They obtained

$$\limsup_{t\to\infty} \left| \frac{P_{\mathcal{E}_a}(t)}{t^{11/8}(\log t)^{3/8}} \right| \le 1237\, a^{1/8},$$

and in fact an explicit though complicated inequality involving several terms in t and a.

We mention a few natural generalizations of the topics described so far: For sharp results on arbitrary origin-centered ellipsoids in dimension $s > 3$, see V. Bentkus and F. Götze [1], and F. Götze [5]. More general problems in higher dimensions are discussed in E. Krätzel's monographs [13] and [14], and in [11]. If we return to $\mathbb{R}^3$ and allow that there are points on $\partial\mathcal{B}$ with Gaussian curvature zero, there are a few partial results available: See E. Krätzel [15], [16], [17], and M. Peter [26]. A general theorem for bodies of rotation was given by the author [25]. This was refined in a special case by E. Krätzel and the author [21].

The objective of the present article is to refine the Vinogradov-Chamizo method which led to (1.4) by a more sophisticated treatment of the exponential sums involved. However, the details turn out to be so complicated that there is little hope to carry out this program for an arbitrary body of rotation $\mathcal{B}_{\mathrm{rot}}$. Instead, we content ourselves with a result for the ellipsoid of rotation given by (1.5). We will add comments on possible further improvements at the end of the paper.

Theorem. *For large real t and fixed $a > 0$, the number $A_{\mathcal{E}_a}(t)$ of lattice points in the ellipsoid*

$$t\mathcal{E}_a \;:\; \frac{u_1^2+u_2^2}{a} + a^2\,u_3^2 \le t^2$$

satisfies the asymptotic

$$A_{\mathcal{E}_a}(t) = \frac{4\pi}{3}t^3 + O\left(t^{679/494+\varepsilon}\right),$$

with $\frac{679}{494} = 1.37449\ldots$ and $\varepsilon > 0$ arbitrary, and the O-constant depends [1] *on a.*

2. Lemmas

Lemma 1. *For integers $k \ge 0$, let as usual $r(k)$ denote the number of pairs $(m_1, m_2) \in \mathbb{Z}^2$ with $m_1^2 + m_2^2 = k$. For large real parameters x, y, and any $\varepsilon > 0$, it then follows that*

$$\begin{aligned} P(x) :&= \sum_{0\le n\le x} r(n) - \pi x \\ &= \frac{1}{\pi}\,x^{1/4}\sum_{1\le k\le y}\frac{r(k)}{k^{3/4}}\,\cos(2\pi\sqrt{kx} - 3\pi/4) + O\left(x^{1/2+\varepsilon}\,y^{-1/2}\right) + O\left(y^{\varepsilon}\right). \end{aligned}$$

[1] This may apply to all constants implicit in the symbols O, $\ll$, $\asymp$ in section 3 through 5.

P r o o f. This is contained in formula (1.9) of A. Ivić [10]. □

Lemma 2. **(a)** *Let $F \in C^4[A,B]$, $G \in C^2[A,B]$, and suppose that, for positive parameters X, Y, Z, we have $1 \le B - A \ll X$ and*

$$F^{(j)} \ll X^{2-j}Y^{-1} \quad \text{for } j=2,3,4,\ F'' \ge c_0 Y^{-1}, \quad G^{(j)} \ll X^{-j}Z \quad \text{for } j=0,1,$$

throughout the interval $[A,B]$, with some constant $c_0 > 0$. Let $\mathcal{J}'$ denote the image of $[A,B]$ under F', and φ the inverse function of F'. Then, with $e(z) := e^{2\pi i z}$ as usual,

$$\sum_{A<k\le B} G(k)\, e(F(k)) = e\left(\frac{1}{8}\right) \sum_{m\in\mathcal{J}'} \frac{G(\varphi(m))}{\sqrt{F''(\varphi(m))}}\, e\left(F(\varphi(m)) - m\varphi(m)\right) \\ + O\left(Z\left(\sqrt{Y} + \log(2 + X/Y)\right)\right).$$

(b) *Let F satisfy the requirements stated, and suppose in addition that $F \in C^7[A,B]$, and that $F^{(3)}$ has at most $O(1)$ sign changes on $[A,B]$. Assume further that there exists a positive parameter Λ such that, for all $v \in \mathcal{J}'$,*

$$\frac{\mathrm{d}^7}{\mathrm{d}v^7}\left(F(\varphi(v)) - v\varphi(v)\right) = -\varphi^{(6)}(v) \asymp \Lambda\,.$$

Then it follows that

$$\sum_{A<k\le B} e(F(k)) \ll \sqrt{Y}\left(1 + \frac{X}{Y}\Lambda^{1/126} + \left(\frac{X}{Y}\right)^{31/32}\Lambda^{-1/126}\right) + \log(2 + X/Y).$$

P r o o f. (a) Transformation formulas of this kind are quite common, though often with worse error terms. This very sharp version can be found as f. (8.47) in the recent monograph [12] of H. Iwaniec and E. Kowalski.
(b) Using clause (a) with $G \equiv 1$ constant, we conclude that

$$\sum_{A<k\le B} e(F(k)) \\ \ll \left| \int_{F'(A)-}^{F'(B)+} F''(\varphi(v))^{-1/2}\, \mathrm{d}\left(\sum_{F'(A)\le m\le v} e(F(\varphi(m)) - m\varphi(m))\right)\right| \\ + \sqrt{Y} + \log(2 + X/Y) \\ \ll \sqrt{Y}\left(1 + \max_{F'(A)\le v\le F'(B)}\left|\sum_{F'(A)\le m\le v} e(F(\varphi(m)) - m\varphi(m))\right|\right) \\ + \log(2 + X/Y).$$

We note that $F'(B) - F'(A) \ll X/Y$, and estimate the last exponential sum by the *"seventh derivative test"*, which is contained, e.g., in Theorem 2.6 in E. Krätzel's monograph [13], p. 34. This readily yields the stated result. [2] $\square$

3. Proof of the Theorem, I: Preparation of the estimate

Employing the usual cut-into-slices approach, we get

$$A_{\mathcal{E}_a}(t) = \sum_{|n| \le \frac{t}{a}} \left(\sum_{n_1^2+n_2^2 \le at^2 - a^3n^2} 1 \right) = \sum_{|n| \le \frac{t}{a}} \Big(\pi \left(at^2 - a^3n^2\right) + P\left(at^2 - a^3n^2\right) \Big). \tag{3.1}$$

By Euler's formula and the second mean-value theorem,

$$\begin{aligned} \pi \sum_{|n| \le \frac{t}{a}} \left(at^2 - a^3n^2\right) &= \pi \int_{-t/a}^{t/a} \left(at^2 - a^3u^2\right) \mathrm{d}u + \int_{-t/a}^{t/a} \psi(u) \, \mathrm{d}\left(at^2 - a^3u^2\right) \\ &= \frac{4\pi}{3} t^3 + O(t), \end{aligned} \tag{3.2}$$

where $\psi(u) = u - [u] - \frac{1}{2}$. Now let W be a parameter with $0 < \frac{t}{a} - W \asymp t^{2/3}$. For $|n| > W$, and also for $n = 0$, we use the classic bound $P(at^2 - a^3n^2) \ll t^{2/3}$. This gives a total error of $O\left(t^{4/3}\right)$. Approximating $P(at^2 - a^3n^2)$ by means of Lemma 1 otherwise, with $x = at^2 - a^3n^2 \ll t^2$ and $y = t^{309/247}$, we obtain

$$\begin{aligned} &\sum_{|n| \le t/a} P\left(at^2 - a^3n^2\right) \\ &= 2 \sum_{1 \le n \le W} P(at^2 - a^3n^2) + O\left(t^{4/3}\right) \\ &= \frac{2}{\pi} \sum_{1 \le k \le t^{309/247}} \frac{r(k)}{k^{3/4}} \left(\sum_{1 \le n \le W} \left(at^2 - a^3n^2\right)^{1/4} \cos\left(2\pi\sqrt{k(at^2 - a^3n^2)} - 3\pi/4\right) \right) \\ &\quad + O\left(t^{679/494+\varepsilon}\right). \end{aligned} \tag{3.3}$$

Keeping k fixed for the moment, we subdivide the range of n into dyadic subintervals $U_j :=](1-2^{-j})\frac{t}{a}, (1-2^{-j-1})\frac{t}{a}]$, $j = 0, 1, \ldots, J$, with J so that $2^J \asymp t^{1/3}$.

[2] The argument seems to contain a flaw for X/Y small. However, in this case it is easy to see that the whole exponential sum on the left is $\ll \sqrt{Y}$.

Then,

$$W := \left(1 - 2^{-J-1}\right) \frac{t}{a} \,. \tag{3.4}$$

Further, let $\mathcal{L}_j := \mathrm{length}(U_j) = 2^{-j-1}\,\frac{t}{a}$, then $\frac{t}{a} - u \asymp \mathcal{L}_j$ for all $u \in U_j$. We shall use Lemma 2 (a) to transform the inner trigonometric sum in (3.3), restricted to any subinterval U_j. Choosing $F(u) = -\sqrt{k}\sqrt{at^2 - a^3u^2}$, $G(u) = (at^2 - a^3u^2)^{1/4}$, we readily infer that

$$F''(u) = \frac{\sqrt{k}\,a^{5/2}\,t^2}{(t^2 - a^2u^2)^{3/2}} \asymp \sqrt{k\,t}\,\mathcal{L}_j^{-3/2}$$

uniformly for $u \in \overline{U_j}$, and accordingly $F^{(3)}(u) \ll \sqrt{k\,t}\,\mathcal{L}_j^{-5/2}$ and $F^{(4)} \ll \sqrt{k\,t}\,\mathcal{L}_j^{-7/2}$. Further, on that range, $G(u) \asymp (t\,\mathcal{L}_j)^{1/4}$, $G'(u) \ll t^{1/4}\,\mathcal{L}_j^{-3/4}$. Hence the conditions of Lemma 2 (a) are fulfilled on each U_j, with the choice of parameters

$$X = \mathcal{L}_j\,, \quad Y = \frac{\mathcal{L}_j^{3/2}}{\sqrt{k\,t}}\,, \quad Z = (t\,\mathcal{L}_j)^{1/4}.$$

The function $\varphi = (F')^{-1}$ of this Lemma is readily computed as

$$\varphi(v) = \frac{t}{a}\left(1 + a^3\,\frac{k}{v^2}\right)^{\frac{-1}{2}} = \varphi_0(v/\sqrt{k}),$$

where $\varphi_0(\xi) = \frac{t}{a}(1 + \frac{a^3}{\xi^2})^{\frac{-1}{2}}$ is independent of k. The error term from Lemma 2 (a), corresponding to U_j, is thus

$$\ll Z(\sqrt{Y} + \log t) \ll \mathcal{L}_j\,k^{-1/4} + (t\,\mathcal{L}_j)^{1/4}\log t\,.$$

Summing this over j and k, we see that these error terms contribute to the right-hand side of (3.3) only

$$\ll \sum_{1 \le k \le t^{309/247}} \frac{r(k)}{k^{3/4}} \left(t\,k^{-1/4} + t^{1/2}\log t\right) \ll t\log t\,.$$

Carrying out the computations required by Lemma 2 (a), and putting together the new exponential sums arising, we arrive at

$$\sum_{|n| \le t/a} P(at^2 - a^3n^2)$$

$$= -\frac{2a^2t}{\pi}\,\Re\left\{ \sum_{\substack{1 \le k \le t^{309/247} \\ m > 0}}^{(*)} \frac{r(k)}{f(k,m)^2}\, e\left(-(t/a)f(k,m)\right)\right\} + O\left(t^{679/494+\varepsilon}\right), \tag{3.5}$$

where

$$f(k,m) := \sqrt{m^2 + a^3 k}\,.$$

Here and throughout the sequel, $\sum^{(*)}$ means that summation is restricted to integer pairs (k,m) such that $\varphi_0(m/\sqrt{k}) \le W$. By (3.4) and a simple Taylor expansion, this last inequality implies that $m \ll \sqrt{k}\, t^{1/6}$.

4. Proof of the Theorem, II: Bounding the exponential sums

It remains to estimate $O(t^\varepsilon)$ sums

$$E^*_{K,M} := \sum_{K<k\le 2K,\ M<m\le 2M}^{(*)} \frac{r(k)}{f(k,m)^2}\, e\left((t/a)f(k,m)\right),$$

where K, M range over dyadic sequences, with $K \ll t^{309/247}$, and $M \ll \sqrt{K}\, t^{1/6}$. We may also assume that K, M are sufficiently large, for otherwise $E^*_{K,M} \ll 1$. Summation by parts gives

$$E^*_{K,M} \ll \frac{1}{K+M^2} \max_{K<k_1\le 2K,\ M<m_1\le 2M} |E_{K,M}(k_1,m_1)|\,, \tag{4.1}$$

where

$$E_{K,M}(k_1,m_1) := \sum_{K<k\le k_1,\ M<m\le m_1}^{(*)} r(k)\, e\left((t/a)f(k,m)\right).$$

We denote by K^*, M^* values of k_1, m_1 for which the maximum in (4.1) is attained, and write E for short instead of $E_{K,M}(K^*,M^*)$. Further, in order to carry out a Weyl step, we introduce another positive integer parameter L with $L \le \frac{1}{2}M$. Obviously,

$$L\,E = \sum_{M-L<m<M^*} \sum_{K<k\le K^*} r(k) \sum_{\ell\in[1,L]\,:\,M<m+\ell\le M^*}^{(*)} e\left((t/a)f(k,m+\ell)\right),$$

where $\overset{(*)}{\sum}$ now refers to $(k, m+\ell)$. Thus, applying Cauchy's inequality twice,

$$L^2|E|^2 \ll M \sum_{M-L<m<M^*} \left| \sum_{K<k\le K^*} r(k) \sum_{\ell\in[1,L]\,:\,M<m+\ell\le M^*}^{(*)} e\left(\frac{t}{a}\, f(k,m+\ell)\right)\right|^2$$
$$\le M \left(\sum_{K<k\le K^*} r^2(k)\right) \sum_{M-L<m<M^*} \sum_{K<k\le K^*}$$
$$\left| \sum_{\ell\in[1,L]\,:\,M<m+\ell\le M^*}^{(*)} e\left(\frac{t}{a}\, f(k,m+\ell)\right)\right|^2$$
$$\ll MK^{1+\varepsilon} \sum_{M-L<m<M^*} \sum_{K<k\le K^*} \sum_{\substack{\ell_1,\ell_2\in[1,L]:\\ m+\ell_1,m+\ell_2\in]M,M^*]}}^{(**)}$$
$$e\left(\frac{t}{a}\left(f(k,m+\ell_1)-f(k,m+\ell_2)\right)\right),$$

where $(**)$ now means that both $\varphi_0((m+\ell_1)/\sqrt{k})$ and $\varphi_0((m+\ell_2)/\sqrt{k})$ are $\le W$. The diagonal terms contribute $\ll M^2LK^{2+\varepsilon}$. In the remaining sum we choose new summation variables $m' = m+\min(\ell_1,\ell_2)$ and $\ell = |\ell_1-\ell_2|$, and obtain

$$\begin{aligned} L^2|E|^2 &\ll M^2LK^{2+\varepsilon} \\ &+ MLK^{1+\varepsilon} \sum_{\substack{\ell\in[1,L[,m'\in]M,M^*]:\\ m'+\ell\in]M,M^*]}} \max_{I\subseteq]K,2K]} \left|\sum_{k\in I} e\left(\frac{t}{a}\left(f(k,m'+\ell)-f(k,m')\right)\right)\right|. \end{aligned} \tag{4.2}$$

where I ranges over all subintervals of $]K,2K]$. It remains to estimate the inner exponential sum $\sum_{k\in I\subseteq]K,2K]} e(F(k))$, where [3]

$$\begin{aligned} F(u) = F_{\ell,m',t}(u) :=& \frac{t}{a}\,\left(f\left(u,m'+\ell\right)-f\left(u,m'\right)\right) \\ =& \frac{t}{a}\,\left(\sqrt{(m'+\ell)^2+a^3u}-\sqrt{m'^2+a^3u}\right), \end{aligned} \tag{4.3}$$

[3]There should be no danger to mix up this new function F with the previous F of section 3.

and the restrictions $\ell \in [1, L[,\ \ m', m'+\ell \in]M, M^*]$ hold. To this end we employ Lemma 2 (b). Since

$$F''(u) = \frac{3}{4}\frac{t}{a}\int\limits_{m'}^{m'+\ell} \frac{a^6\,\xi\ \mathrm{d}\xi}{(a^3u+\xi^2)^{5/2}},$$

and similarly for $F^{(3)},\ F^{(4)}$, the requirements on $F'',\ F^{(3)},\ F^{(4)}$ are satisfied with

$$X = K, \quad Y = \frac{(M^2+K)^{5/2}}{\ell M t}.$$

To verify the condition involving Λ, we first note that, for quite general F and φ in the sense of Lemma 2 (b),

$$\begin{aligned}
&\frac{\mathrm{d}^7}{\mathrm{d}v^7}\left(F(\varphi(v)) - v\varphi(v)\right) = -\varphi^{(6)}(v)\\
&= -(F'')^{-11}\Big(-945(F^{(3)})^5 + 1260F''(F^{(3)})^3F^{(4)} - 210(F'')^2(F^{(3)})^2F^{(5)}\\
&+7(F'')^2F^{(3)}\left(-40(F^{(4)})^2 + 3F''F^{(6)}\right) + (F'')^3\Big(35F^{(4)}F^{(5)} - F''F^{(7)}\Big)\Big)\Big|_{u=\varphi(v)}.
\end{aligned}$$

While the first equality is obvious, the second one follows by an involved but straightforward computation, starting from $\varphi'(v) = (F''(\varphi(v))^{-1}$, and preferably using a computer algebra system like *Mathematica*. Specifying F according to (4.3), we obtain further, supported again by *Mathematica*,

$$\begin{aligned}
&\frac{\mathrm{d}^7}{\mathrm{d}v^7}\left(F(\varphi(v)) - v\varphi(v)\right) = -\varphi^{(6)}(v)\\
&= \quad - \left.\frac{(UV)^4\,\mathcal{P}_{20}^{+}(\sqrt{U},\sqrt{V})}{a^{15}t^6(\sqrt{U}-\sqrt{V})^6(U+\sqrt{UV}+V)^{11}}\right|_{\substack{U=(m'+\ell)^2+a^3\varphi(v)\\ V=m'^2+a^3\varphi(v)}},
\end{aligned}$$

where $\mathcal{P}_{20}^{+}$ is an explicit homogeneous polynomial in two variables of degree 20 whose coefficients are all positive integers. Further, by the restrictions imposed,

$$U \asymp V \asymp M^2 + K\,, \quad \text{and} \quad \sqrt{U} - \sqrt{V} \asymp \frac{M\,\ell}{\sqrt{M^2+K}}\,.$$

Hence the last condition of Lemma 2 (b) is satisfied with $\Lambda = (M\ell t)^{-6}(M^2 + K)^{10}$. Applying this Lemma, we obtain

$$\sum_{k \in I \subseteq]K,2K]} e\left(\frac{t}{a}\left(f(k, m' + \ell) - f(k, m')\right)\right)$$
$$\ll \frac{(M^2+K)^{5/4}}{(M\ell t)^{1/2}} \left(1 + \frac{\ell M K t}{(M^2+K)^{5/2}} \left((M\ell t)^{-6}(M^2+K)^{10}\right)^{1/126} \right.$$
$$\left. + \left(\frac{\ell M K t}{(M^2+K)^{5/2}}\right)^{31/32} \left((M\ell t)^{-6}(M^2+K)^{10}\right)^{-1/126}\right) + t^{\varepsilon}.$$

We use this in (4.2), carry out the summation over ℓ and m', and divide by L^2 to arrive at

$$|E|^2 \ll \frac{K^2 M^2}{L} t^{\varepsilon} + \frac{K^2 (Lt)^{19/42} M^{103/42} t^{\varepsilon}}{(M^2+K)^{295/252}} + \frac{K^{63/32} (Lt)^{347/672} M^{1691/672} t^{\varepsilon}}{(M^2+K)^{5045/4032}}$$
$$+ K M^2 t^{2\varepsilon} + \frac{K M^{3/2} (M^2+K)^{5/4}}{\sqrt{Lt}} t^{\varepsilon}.$$

Taking the square root (term by term) and going back to (4.1), we conclude that

$$E^*_{K,M} \ll \frac{K M t^{\varepsilon/2}}{\sqrt{L}\,(M^2+K)} + \frac{K (Lt)^{19/84} M^{103/84} t^{\varepsilon/2}}{(M^2+K)^{799/504}}$$
$$+ \frac{K^{63/64} (Lt)^{347/1344} M^{1691/1344} t^{\varepsilon/2}}{(M^2+K)^{13109/8064}}$$
$$+ \frac{\sqrt{K}\, M t^{\varepsilon}}{M^2+K} + \frac{\sqrt{K}\, M^{3/4} t^{\varepsilon/2}}{(Lt)^{1/4} (M^2+K)^{3/8}}\,. \tag{4.4}$$

Here the first two terms on the right hand side are of the same size if L equals

$$L_0 := \frac{(M^2+K)^{295/366}}{(Mt)^{19/61}}\,.$$

Hence we define the positive integer L by

$$L := \begin{cases} [L_0] & \text{if } 1 \le L_0 \le \frac{1}{2}M, \\ [\frac{1}{2}M] & \text{if } L_0 > \frac{1}{2}M. \end{cases}$$

Then always $1 \le L \le \frac{1}{2}M$ as assumed, provided that $L_0 \ge 1$. The exceptional case that $L_0 < 1$ will be treated differently, by a trivial bound. We altogether have to distinguish three cases.

Case 1. $1 \le L_0 \le \frac{1}{2}M$. Choosing $L = [L_0]$, we deduce from (4.4) that

$$E^*_{K,M} \ll \frac{KM^{141/122}\,t^{19/122}t^{\varepsilon/2}}{(M^2+K)^{1027/732}} + \frac{K^{63/64}M^{2299/1952}\,t^{347/1952}t^{\varepsilon/2}}{(M^2+K)^{2767/1952}} + \frac{\sqrt{K}\,Mt^{\varepsilon}}{M^2+K} + \frac{\sqrt{K}\,M^{101/122}t^{\varepsilon/2}}{(M^2+K)^{211/366}\,t^{21/122}}.$$

We now estimate M throughout by $(M^2+K)^{1/2}$, then $(M^2+K)^{-1}$ by K^{-1}. (All exponents happen to have the suitable sign.) This gives

$$E^*_{K,M} \ll K^{32/183}\,t^{19/122+\varepsilon} + K^{19/122}\,t^{347/1952+\varepsilon} + K^{247/732}\,t^{-21/122+\varepsilon} + t^{\varepsilon}.$$

Recalling finally that $K \ll t^{309/247}$, we readily deduce that

$$E^*_{K,M} \ll t^{185/494+\varepsilon} \tag{4.5}$$

in this case.

Case 2. $L_0 > \frac{1}{2}M$. Explicitly, this means that

$$(M^2+K)^{295/366} \gg M^{80/61}\,t^{19/61}. \tag{4.6}$$

It readily follows that $M^2 \le K$, since otherwise $M^2+K \ll M^2$, and (4.6) would imply that $M \gg t^{57/55}$, which contradicts $M \ll \sqrt{K}\,t^{1/6} \ll t^{587/741}$. We choose $L = [\frac{1}{2}M]$ in (4.4), and simplify this expression, using the relations $M^2+K \asymp K$, $M \ll K^{59/96}\,t^{-19/80}$ (from (4.6)), and $K \ll t^{309/247}$. This ultimately yields

$$E^*_{K,M} \ll t^{20999/79040+\varepsilon},$$

which is even sharper than (4.5).

Case 3. $L_0 < 1$. Explicitly, this means that $M^{295/57} < (M^2+K)^{295/114} < Mt$, hence $M < t^{57/238}$. Thus, by a trivial estimate,

$$E^*_{K,M} \ll \frac{KM}{M^2+K} < M < t^{57/238},$$

which is much sharper than (4.5). Therefore, (4.5) is true in all cases. Recalling (3.5), (3.1), and (3.2), we complete the proof of our Theorem.

5. Comments on possible refinements

Thinking about possibilities to improve upon the result just established, let us assume that we follow the present approach until formula (4.2), and seek for sharper bounds for the innermost sum $\sum_{k\in I\subseteq]K,2K]} e(F(k))$. For the purpose of a heuristic discussion, let us argue as if the function F defined in (4.3) belonged to the class of functions to which the *method of exponent pairs* applies. (For an enlightening exposition of the latter, see E. Krätzel [13], ch. 2.1.4.) If we estimate this inner sum by means of any exponent pair (α, β), a straightforward repetition of the above argument yields the final result

$$P_{\mathcal{E}_a}(t) = O\left(t^{\lambda(\alpha,\beta)+\varepsilon}\right), \qquad \lambda(\alpha,\beta) = \frac{3\alpha + 4\beta + 2}{2(\alpha+\beta+1)}.$$

Using the easiest nontrivial exponent pair $(\frac{1}{2}, \frac{1}{2})$, one obtains Chamizo's $O\left(t^{11/8+\varepsilon}\right)$, while the pair $BA^5(\frac{1}{2}, \frac{1}{2}) = (\frac{19}{42}, \frac{32}{63})$ leads to the bound $O\left(t^{679/494+\varepsilon}\right)$ of the present paper. Increasing here the exponent 5 of A would weaken the estimate. Adding further A-processes before the "B" leads to overwhelming technical difficulties when trying an exact calculation. Therefore, as long as the classic van der Corput's method for one-dimensional exponential sums is concerned, our choice seems to be optimal in a sense.

However, M. Huxley by his "Discrete Hardy-Littlewood Method", as exposed in his monograph [8], established an entirely new approach to the estimation of exponential sums. In its sharpest form [9], his deep theory tells us that $\left(\frac{32}{205} + \varepsilon', \frac{1}{2} + \frac{32}{205} + \varepsilon'\right)$ is an exponent pair for every $\varepsilon' > 0$. Since this pair is close to $\left(\frac{1}{6}, \frac{2}{3}\right) = A\left(\frac{1}{2}, \frac{1}{2}\right)$, by analogy to the choice above we are led to consider $BA^4\left(\frac{32}{205} + \varepsilon', \frac{1}{2} + \frac{32}{205} + \varepsilon'\right) = \left(\frac{3843}{8480} + \varepsilon'', \frac{269}{530} + \varepsilon''\right)$, with $\varepsilon'' > 0$ arbitrarily small. Using this last pair to bound $\sum_{k \in I \subseteq]K,2K]} e(F(k))$, we finally would arrive at

$$P_{\mathcal{E}_a}(t) = O\left(t^{45705/33254+\varepsilon}\right), \tag{5.1}$$

which is slightly better than our Theorem. Trying to make this argument exact, we are facing two difficulties: Firstly, we have to verify the non-vanishing of another higher derivative. This can be accomplished again by means of *Mathematica*. More serious is the lack of a direct analogue to Theorem 2.6 in E. Krätzel [13]. It would be necessary to derive such a result from Huxley's bound in [9], applying a differencing lemma for exponential sums four times. However, we readily calculate that our present improvement amounts to $\frac{11}{8} - \frac{679}{494} = \frac{1}{1976} \approx 5.06 \times 10^{-4}$, while this further refinement would only be $\frac{679}{494} - \frac{45705}{33254} = \frac{23}{315913} \approx 7.28 \times 10^{-5}$. Therefore, we do not enter into the details of a proof of (5.1) in the scope of this paper.

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