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A COMPARISON THEOREM FOR MATRIX LIMITATION METHODS WITH APPLICATIONS

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ABSTRACT. A comparison Theorem for matrix limitation methods is proved and the following applications are given:
- new results concerning the comparison between weighted densities generated by different weights;
- new results concerning the comparison between weighted densities of a set $E \subseteq \mathbb{N}^*$ and those of its transformed set $\pi(E)$, where π is a given injective function $\mathbb{N}^* \to \mathbb{N}^*$. In particular, a new class of permutations preserving the asymptotic density is identified.

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1. Introduction

The upper and lower asymptotic densities of a subset A of $\mathbb{N}^*$ (where $\mathbb{N}^*$ is the set of strictly positive integers) are defined respectively by

$$\limsup_{n\to\infty} \frac{\mathrm{card}\{k \in A, 1 \le k \le n\}}{n};$$
$$\liminf_{n\to\infty} \frac{\mathrm{card}\{k \in A, 1 \le k \le n\}}{n}.$$

Notice that, setting

$$\mathrm{card}\{k \in A, 1 \le k \le n\} = \sum_{\substack{1\le k\le n\\ k\in A}} a_k; \qquad n = \sum_{1\le k\le n} a_k,$$

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where $a_k = 1$ for every integer k, the above ratio takes the form

$$\frac{\sum_{\substack{1\le k\le n\\ k\in A}} a_k}{\sum_{k=1}^{n} a_k},$$

and the corresponding upper and lower densities reveal now their nature of "weighted" densities with constant "weights" equal to 1.

The notion of weighted upper and lower densities of a subset A of $\mathbb{N}^*$ (with general weights (a_k)) was given for the first time in [Rohrbach et al.15] and later in [Alexander 1]; it is recalled at the beginning of Section 3. It was used also later in [Giuliano Antonini 4]. The problem of comparing two weighted densities generated by different sequences of weights is quite natural. In [Giuliano Antonini 5] and [Fuchs et al. 3] some answers are given. In the present paper we address the question from another point of view, as explained below.

As is well known, weighted densities of subsets of $\mathbb{N}^*$ can be obtained by using suitable limitation methods generated by Riesz matrices; see the beginning of Section 3 for definitions and details.

Motivated by the preceding remark, we prove in the present paper a comparison Theorem for matrix limitation methods and give some new applications in the study of weighted densities.

Classical Theorems concerning the comparison of different matrix limitation methods typically only treat the case of convergent sequences, and their assumptions bear on the modulus of some quantities related to the elements of the matrices involved, say $\mathbf{U}$ and $\mathbf{V}$ to fix ideas (see [Petersen 12] or [Peyerimhoff 13] as references on the topic). In the present paper we consider the more general case of bounded sequences. This generalization turns out to be important as can be seen in our applications (see Sections 3 and 5).

More precisely, we prove a general statement (Theorem 2.7) with assumptions less stringent than the usual ones: in fact, we need to control *only a subset* of the whole set of elements of the product matrix $\mathbf{U}\mathbf{V}^{-1}$ in order to obtain relations (2.8) and (2.9) (i.e., the comparison between the two limitation methods). Note that (2.8) and (2.9) are assertions concerning the lim sup and the lim inf of the involved sequences (and not the limits, which need not exist). This is done in Section 2. In the rest of the paper we give the announced applications of the general comparison Theorem 2.7. The applications are in two directions:

(i) we prove many new results extending weighted density (Sections 3 and 4, Theorem 3.9, Corollaries 3.10 and 3.11);

(ii) we study the problem of the comparison between the weighted densities of a set $E \subseteq \mathbb{N}^*$ and of its transformed set $\pi(E)$, for a given injective function $\pi : \mathbb{N}^* \to \mathbb{N}^*$ (Section 5, Theorem 5.4).

In particular, we identify a class of permutations of $\mathbb{N}^*$ that preserve weighted densities (see Remark 5.7 and Corollary 5.8). As far as we know, results in this direction have been previously obtained only for the case of asymptotic density (see [Obata 10], where, in particular, the so called *Lévy group* is considered); nevertheless, even in this case, the class of permutations of Corollary 5.8 seems to be new.

Moreover our Theorem 5.4 generalizes, in the sense explained in Remark 5.6, a recent result obtained in [Nathanson et al. 9].

A preliminary version of the present paper was presented in the preprint [Giuliano Antonini et al. 7].

Notation 1.1. Let (a_n) and (b_n) be two sequences of real numbers and $\mathcal{C}$ an infinite subset of $\mathbb{N}^*$. Throughout the whole paper the notation $a_n \sim b_n$, $n \to \infty$, $n \in \mathcal{C}$ means that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{C}}} \frac{a_n}{b_n} = 1.$$

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2. Preliminary definitions and the comparison Theorem

Let $\mathbf{U} = (u_{n,k})_{n,k}$ and $\mathbf{V} = (v_{n,k})_{n,k}$ be two infinite matrices; let $\epsilon = (\epsilon_n)_n$ be a sequence of numbers and $\mathbf{U}\epsilon = \eta = (\eta_n)_n$ and $\mathbf{V}\epsilon = \xi = (\xi_n)_n$ the two sequences defined respectively by

$$\eta_n = (\mathbf{U}\epsilon)_n = \sum_{k=1}^{\infty} u_{n,k}\epsilon_k, \qquad \xi_n = (\mathbf{V}\epsilon)_n = \sum_{k=1}^{\infty} v_{n,k}\epsilon_k \tag{2.1}$$

(here and in the sequel we assume that all the series considered are convergent).

Remark 2.2. In [Petersen 12], p. 4, the following definition is given:
Definition. A *limitation method* is a linear transformation defined on the set of all sequences of real numbers which maps bounded sequences into bounded sequences.

Clearly the linear transformation $\epsilon \mapsto \mathbf{U}\epsilon$ (resp. $\epsilon \mapsto \mathbf{V}\epsilon$) is a limitation method if and only if

$$\sup_n \sum_{k=1}^{\infty} |u_{n,k}| < +\infty \qquad \text{(resp. } \sup_n \sum_{k=1}^{\infty} |v_{n,k}| < +\infty).$$

Put

$$\begin{cases} \underline{l} &= \liminf_{n\to\infty} \eta_n, \\ \overline{l} &= \limsup_{n\to\infty} \eta_n, \end{cases} \qquad \begin{cases} \underline{m} &= \liminf_{n\to\infty} \xi_n, \\ \overline{m} &= \limsup_{n\to\infty} \xi_n. \end{cases} \tag{2.3}$$

Definition 2.4. Given the infinite matrix $\mathbf{U}$, the sequence $\epsilon = (\epsilon_n)_n$ is said to be $\mathbf{U}$–*limitable* if the limit

$$l = \lim_{n\to\infty} \eta_n = (\mathbf{U}\epsilon)_n$$

exists and is finite (i.e. $\underline{l} = \overline{l} \in \mathbb{R}$).

In order to state our comparison Theorem, we need another definition.

Definition 2.5. Let α, β be two real numbers, with $\alpha \le \beta$. We say that the infinite matrix $\mathbf{U} = (u_{n,k})$ is an (α, β)–*matrix* if for every sufficiently large n the series $\sum_{k=1}^{\infty} u_{n,k}$ converges and

$$\liminf_{n\to\infty} \sum_{k=1}^{\infty} u_{n,k} = \alpha, \qquad \limsup_{n\to\infty} \sum_{k=1}^{\infty} u_{n,k} = \beta.$$

An (α, β)–matrix $\mathbf{U}$ will be said to be *regular* if

$$\lim_{n\to\infty} u_{n,k} = 0, \qquad k = 1, 2, \dots.$$

Remark 2.6. We recall the classical definition of the Toeplitz matrix (see [Kuipers and Niederreiter 8], p. 60 ff. as a reference):

Definition. The infinite matrix $\mathbf{U} = (u_{n,k})$ is called a *Toeplitz matrix* if $u_{n,k} \ge 0$ for all n and k and if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} u_{n,k} = 1.$$

Hence, a Toeplitz matrix is a (1,1)–matrix, according to Definition 2.5. Note that in our Definition 2.5 we do not assume that $u_{n,k} \ge 0$ for all n and k. This extension will be crucial in the application of Section 5 (see Remark 5.18).

For every subset E of $\mathbb{N}^*$, we denote by 1_E the *indicator function* of E, i.e.

$$1_E(k) = \begin{cases} 1 & \text{if} k \in E, \\ 0 & \text{if} k \in E^c. \end{cases}$$

The announced comparison Theorem is the following

Theorem 2.7. *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be two infinite matrices and* η, ξ *be defined as in* (2.1). *Assume that* $\mathbf{U}$ *is invertible with inverse* $\mathbf{U}^{-1}$ *and suppose that the infinite matrix*

$$\mathbf{V}\mathbf{U}^{-1} = \mathbf{W} = (w_{n,k})_{n,k}$$

is a regular (α, β)*–matrix, with* $\alpha \geq 0$. *For every integer* n, *consider the subset* $\mathcal{W}_n$ *of* $\mathbb{N}^*$ *defined by*

$$\mathcal{W}_n = \{k \in \mathbb{N}^* : w_{n,k} \leq 0\},$$

and assume that

$$\sigma_{\mathbf{W}} := \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} |w_{n,k}| 1_{\mathcal{W}_n}(k) \right) < +\infty.$$

Assume that $\underline{l}$, $\bar{l}$, $\underline{m}$ *and* $\overline{m}$ *(defined in* (2.3)*) are finite. Then*
(i) if $0 \leq \underline{l} \leq \bar{l}$,

$$\sigma_{\mathbf{W}}(\underline{l} - \bar{l}) + \alpha\, \underline{l} \leq \underline{m} \leq \overline{m} \leq \sigma_{\mathbf{W}}(\bar{l} - \underline{l}) + \beta\, \bar{l}; \tag{2.8}$$

(ii) if $\underline{l} \leq \bar{l} < 0$,

$$\sigma_{\mathbf{W}}(\underline{l} - \bar{l}) + \beta\, \underline{l} \leq \underline{m} \leq \overline{m} \leq \sigma_{\mathbf{W}}(\bar{l} - \underline{l}) + \alpha\, \bar{l}; \tag{2.9}$$

Remark 2.10. The assumption $\alpha \geq 0$ is strictly weaker than the assumption that $w_{n,k} \geq 0$ for each pair (n, k). A concrete example is given by the situation considered in Theorem 5.4 below: see the first formula in (5.16) and formulas (5.17).

Proof. We can write $\epsilon = \mathbf{U}^{-1}\eta$, hence $\xi = \mathbf{V}\mathbf{U}^{-1}\eta = \mathbf{W}\eta$. This means that, for every n, we have the relation

$$\xi_n = \sum_{k=1}^{\infty} w_{n,k}\eta_k. \tag{2.11}$$

(i) Assume that $0 \leq \underline{l} \leq \bar{l}$ and fix $\delta > 0$. By definition, there exists an n_0 such that, for $n > n_0$, we have

$$\underline{l} - \delta \leq \eta_n \leq \bar{l} + \delta.$$

In order to simplify the notations, we put $\mathcal{D}_n = \mathcal{W}_n \cap [n_0 + 1, +\infty)$ and $\mathcal{E}_n = \mathcal{W}_n^c \cap [n_0 + 1, +\infty)$; hence the right hand side of (2.11) is less than or equal to

$$\begin{aligned}
&\sum_{k=1}^{n_0} w_{n,k}\eta_k + (\bar{l}+\delta)\sum_{k\in\mathcal{E}_n} w_{n,k} + (\underline{l}-\delta)\sum_{k\in\mathcal{D}_n} w_{n,k}\\
&\quad= \sum_{k=1}^{n_0} w_{n,k}\eta_k + (\bar{l}+\delta)\left(\sum_{k=n_0+1}^{\infty} w_{n,k} - \sum_{k\in\mathcal{D}_n} w_{n,k}\right) + (\underline{l}-\delta)\sum_{k\in\mathcal{D}_n} w_{n,k}\\
&\quad= \sum_{k=1}^{n_0} w_{n,k}(\eta_k - \bar{l}-\delta) + (\bar{l}+\delta)\sum_{k=1}^{\infty} w_{n,k} - (\bar{l}-\underline{l}+2\delta)\sum_{k\in\mathcal{D}_n} w_{n,k}\\
&\quad= \sum_{k=1}^{n_0} w_{n,k}(\eta_k - \bar{l}-\delta) + (\bar{l}+\delta)\sum_{k=1}^{\infty} w_{n,k} -\\
&\qquad - (\bar{l}-\underline{l}+2\delta)\left\{\sum_{k=1}^{\infty} w_{n,k}1_{\mathcal{W}_n(k)} - \sum_{k=1}^{n_0} w_{n,k}1_{\mathcal{W}_n}(k)\right\}.
\end{aligned}$$

The first and last sums above tend to 0 as $n\to\infty$ since $\mathbf{W}$ is regular: in fact, for the first sum, we observe that

$$\left|\sum_{k=1}^{n_0} w_{n,k}(\eta_k - \bar{l}-\delta)\right| \le \left(\sup_{1\le k\le n_0}|\eta_k| + \bar{l}+\delta\right)\sum_{k=1}^{n_0}|w_{n,k}|;$$

for the last sum, we notice that $|w_{n,k}1_{\mathcal{W}_n}(k)| \le |w_{n,k}|$.

The second sum above has a lim sup equal to β since $\mathbf{W}$ is an (α,β)–matrix. Since $\bar{l}+\delta > 0$ and $\bar{l}-\underline{l}+2\delta > 0$, the last inequality of (2.8) now follows immediately by the definition of $\sigma_{\mathbf{W}}$ and the arbitrariness of δ.

The first inequality of (2.8) (concerning the lim inf) is proved analogously if $\underline{l} > 0$: we fix δ, with $0<\delta<\underline{l}$ and, by interchanging $\bar{l}+\delta$ with $\underline{l}-\delta$ and vice versa and by repeating the above calculation, we get the lower bound

$$\begin{aligned}
&\sum_{k=1}^{n_0} w_{n,k}(\eta_k - \underline{l}+\delta) + (\underline{l}-\delta)\sum_{k=1}^{\infty} w_{n,k} -\\
&\qquad - (\underline{l}-\bar{l}-2\delta)\left\{\sum_{k=1}^{\infty} w_{n,k}1_{\mathcal{W}_n(k)} - \sum_{k=1}^{n_0} w_{n,k}1_{\mathcal{W}_n}(k)\right\},
\end{aligned}$$

in which we can pass to the limit as $n\to\infty$, concluding again using the arbitrariness of δ.

In the case $\underline{l}=0$, we get the lower bound

$$\sum_{k=1}^{n_0} w_{n,k}(\eta_k+\delta) - \delta\sum_{k=1}^{\infty} w_{n,k} + (\bar{l}+2\delta)\cdot\left\{\sum_{k=1}^{\infty} w_{n,k}1_{\mathcal{W}_n(k)} - \sum_{k=1}^{n_0} w_{n,k}1_{\mathcal{W}_n}(k)\right\},$$

which, by passing to the limit as $n \to \infty$, becomes

$$-\delta\,\beta - \sigma_{\mathbf{W}}(\bar{l} + 2\delta)$$

whence the first of inequalities (2.8) follows, once again by the arbitrariness of δ.

(ii) If $\underline{l} \le \bar{l} < 0$ we can apply the argument used for (i) to the sequence $\tilde{\epsilon} = (\tilde{\epsilon}_n)_n$, where $\tilde{\epsilon}_n = -\epsilon_n$. Then a change of sign gives the inequalities (2.9). □

The following two Corollaries are obvious:

Corollary 2.12. *Assume that $\sigma_{\mathbf{W}} < +\infty$ and $\alpha = \beta$. Then every $\mathbf{U}$–limitable sequence is also $\mathbf{V}$–limitable, and the original limit is multiplied by α. (In particular, if $\mathbf{W}$ is Toeplitz, it remains unchanged).*

Corollary 2.13. *Assume that $\sigma_{\mathbf{W}} = 0$ (which means that $w_{n,k} \ge 0$ for all n and k). Then the following inequalities hold*

$$\alpha\,\underline{l} \le \underline{m} \le \overline{m} \le \beta\,\bar{l}.$$

Example 2.14. We begin with a definition:

Definition 2.15. Let (a_n) be a sequence of positive numbers. For every $k, n \in \mathbb{N}^*$ put

$$a_{n,k} = \begin{cases} a_k/S_n & \text{for } k \le n \\ 0 & \text{for } k > n, \end{cases}$$

where

$$S_n = \sum_{k=1}^{n} a_k.$$

The infinite matrix $\mathbf{A} = (a_{n,k})$ is called a *Riesz matrix.*

It is easy to see that the Riesz matrix $\mathbf{A}$ is invertible and its inverse $\mathbf{A}^{-1} = (\tilde{a}_{h,k})$ is given by

$$\tilde{a}_{h,k} = \begin{cases} S_k/a_k & \text{if } h = k, \\ -S_k/a_{k+1} & \text{if } h = k+1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Let $\mathbf{V}$ be any regular (α, β)–matrix and, in order to maintain the notations of Theorem 2.7, put $\mathbf{U} = \mathbf{A}$. Then the matrix $\mathbf{W} = \mathbf{V}\mathbf{U}^{-1}$ is regular and (α, β); in fact in this case we have

$$w_{n,k} = \sum_{h=1}^{\infty} v_{n,h}\tilde{a}_{h,k} = v_{n,k}\frac{S_k}{a_k} - v_{n,k+1}\frac{S_k}{a_{k+1}},$$

so that $\mathbf{W}$ is regular since $\mathbf{V}$ is so. Moreover, by summing over k in the above relation we get

$$\sum_{k=1}^{\infty} w_{n,k} = \sum_{k=1}^{\infty} v_{n,k}\frac{S_k}{a_k} - \sum_{k=2}^{\infty} v_{n,k}\frac{S_{k-1}}{a_k} = v_{n,1} + \sum_{k=2}^{\infty} v_{n,k}\frac{S_k - S_{k-1}}{a_k} = \sum_{k=1}^{\infty} v_{n,k},$$

and this shows that $\mathbf{W}$ is (α,β) since $\mathbf{V}$ is so.

Note that in this case we have

$$\mathcal{W}_n = \left\{ k \in \mathbb{N}^* : \frac{v_{n,k}}{a_k} - \frac{v_{n,k+1}}{a_{k+1}} \leq 0 \right\}.$$

Remark 2.17. In particular, all the above considerations hold true if $\mathbf{V}$ is another Riesz matrix (say $\mathbf{B}$, defined by the sequence (b_n)); if this is the case, we have $\alpha = \beta = 1$ and

$$\mathcal{W}_n = \mathcal{W} = \left\{ k \in \mathbb{N}^* : \frac{b_k}{a_k} - \frac{b_{k+1}}{a_{k+1}} \leq 0 \right\}.$$

3. Applications to the extension of a weighted density

In this Section we present the first application of Theorem 2.7. We begin by describing the framework of our discussion.

Let $a = (a_n)$ and $b = (b_n)$ be two sequences of non negative numbers; put

$$S_n := \sum_{k=1}^{n} a_k, \qquad T_n := \sum_{k=1}^{n} b_k$$

and, for every subset E of $\mathbb{N}^*$ and every integer n,

$$s_n(E) := \sum_{\substack{1\leq k\leq n \\ k\in E}} a_k = \sum_{k=1}^{n} a_k 1_E(k), \quad t_n(E) := \sum_{\substack{1\leq k\leq n \\ k\in E}} b_k = \sum_{k=1}^{n} b_k 1_E(k). \tag{3.1}$$

We assume that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n = +\infty.$$

Denote by $\overline{d}_a(E)$ (resp. $\underline{d}_a(E)$) the upper (resp. lower) weighted density of E with respect to the defining sequence (a_n):

$$\overline{d}_a(E) := \limsup_{n\to\infty} \frac{s_n(E)}{S_n}, \qquad \underline{d}_a(E) := \liminf_{n\to\infty} \frac{s_n(E)}{S_n}. \tag{3.2}$$

If $\overline{d}_a(E) = \underline{d}_a(E) = d_a(E)$ we say that E has a d_a-density, equal to $d_a(E)$. The symbols $\overline{d}_b(E)$, $\underline{d}_b(E)$ and $d_b(E)$ will have analogous meanings, with respect to the defining sequence (b_n).

Looking at the present situation in the setting of infinite matrices (see Section 2), we see that the sequences

$$\left(\frac{s_n(E)}{S_n}\right)_n \quad \text{and} \quad \left(\frac{t_n(E)}{T_n}\right)_n$$

are obtained by transforming the initial sequence $(1_E(n))_n$ by means of the limitation methods defined by the Riesz matrices $\mathbf{A} = (a_{n,k})$ and $\mathbf{B} = (b_{n,k})$, where

$$a_{n,k} = \begin{cases} a_k/S_n & \text{for } k \leq n, \\ 0 & \text{for } k > n, \end{cases} \quad \text{and} \quad b_{n,k} = \begin{cases} b_k/T_n & \text{for } k \leq n, \\ 0 & \text{for } k > n, \end{cases}$$

respectively (for the definition of a Riesz matrix, see 2.15).

Going back to the notion of weighted density, we give three definitions.

Definition 3.3. We say that the pair $(\underline{d}_b, \overline{d}_b)$ *extends* the pair $(\underline{d}_a, \overline{d}_a)$ if, for any subset E of $\mathbb{N}^*$ the following relations hold

$$\underline{d}_a(E) \leq \underline{d}_b(E) \leq \overline{d}_b(E) \leq \overline{d}_a(E).$$

Definition 3.4. We say that d_b *extends* d_a if every set $E \subseteq \mathbb{N}^*$ which has d_a-density possesses also a d_b-density and

$$d_b(E) = d_a(E).$$

Remark 3.5. Of course, if $(\underline{d}_b, \overline{d}_b)$ extends $(\underline{d}_a, \overline{d}_a)$, then d_b extends d_a.

Definition 3.6. We say that d_a and d_b are *equivalent* if d_a is an extension of d_b and conversely (in the sense of Definition 3.4).

Assume now that $a_n \neq 0$ for all n and define the sequence (c_n) by

$$c_n := \frac{b_n}{a_n}, \qquad \forall\, n \in \mathbb{N}^*.$$

It is proved in the paper [Rajagopal 14] that, if (c_n) is not increasing, then $(\underline{d}_b, \overline{d}_b)$ extends $(\underline{d}_a, \overline{d}_a)$; by applying our comparison Theorem 2.7 to the matrices $\mathbf{U} = \mathbf{A}$ and $\mathbf{V} = \mathbf{B}$, we are now in a position to prove a much more general result; following Remark 2.17, we define the sets

$$\mathcal{C} = \{n \in \mathbb{N}^* : c_n - c_{n+1} \leq 0\}, \qquad \mathcal{D} = \{n \in \mathbb{N}^* : c_n - c_{n+1} \geq 0\}$$

and, in view of Theorem 2.7, we are interested in the quantities

$$\sigma_1 \quad := \quad \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} S_k(c_{k+1} - c_k) 1_{\mathcal{C}}(k)}{T_n}, \tag{3.7}$$

$$\sigma_2 \quad := \quad \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} T_k(\tilde{c}_{k+1} - \tilde{c}_k) 1_{\mathcal{D}}(k)}{S_n}, \tag{3.8}$$

where we have put

$$\tilde{c}_n := \frac{1}{c_n} = \frac{a_n}{b_n},$$

provided that $b_n \neq 0$ for every n. Observe that both $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ (since $c_{n+1} \geq c_n$ for every $n \in \mathcal{C}$ and $\tilde{c}_n \leq \tilde{c}_{n+1}$ for every $n \in \mathcal{D}$).

Recalling that $\alpha = \beta = 1$ (see Remark 2.17), we get from Theorem 2.7

Theorem 3.9. *(i) Assume that $\sigma_1 < +\infty$. Then, for every subset E of $\mathbb{N}^*$, we have the relations*

$$\sigma_1\big(\underline{d}_a(E) - \overline{d}_a(E)\big) + \underline{d}_a(E) \leq \underline{d}_b(E) \leq \overline{d}_b(E) \leq \sigma_1\big(\overline{d}_a(E) - \underline{d}_a(E)\big) + \overline{d}_a(E).$$

(ii) Assume that $\sigma_2 < +\infty$. Then, for every subset E of $\mathbb{N}^*$, we have the relations

$$\sigma_2\big(\underline{d}_b(E) - \overline{d}_b(E)\big) + \underline{d}_b(E) \leq \underline{d}_a(E) \leq \overline{d}_a(E) \leq \sigma_2\big(\overline{d}_b(E) - \underline{d}_b(E)\big) + \overline{d}_b(E).$$

Corollaries 2.12 and 2.13 now become:

Corollary 3.10. *Assume as above that $\sigma_1 < +\infty$. Then d_b extends d_a. A dual statement holds (of course with the roles of d_a and d_b interchanged) if $\sigma_2 < +\infty$.*

Corollary 3.11. *Assume that $\sigma_1 = 0$. Then $(\underline{d}_b, \overline{d}_b)$ extends $(\underline{d}_a, \overline{d}_a)$. Once more, a dual statement holds (with the roles of $(\underline{d}_a, \overline{d}_a)$ and $(\underline{d}_b, \overline{d}_b)$ interchanged) if $\sigma_2 = 0$.*

4. Some examples

This Section is concerned with the study of sequences on the right hand sides of (3.7) and (3.8). The aim is to investigate some situations in which $\sigma_1 < +\infty$ or $\sigma_1 = 0$ (resp. $\sigma_2 < +\infty$ or $\sigma_2 = 0$) in order to apply Corollary 3.10 or Corollary 3.11.

We shall perform the calculations only for σ_1, since they are analogous for σ_2.

We first introduce two new quantities (τ_1 and τ_2 below) which will be useful in our discussions (see Propositions 4.3 and 4.4). For this purpose, observe that, by summing and subtracting the term $1_{\mathcal{C}}(k) c_k S_{k-1}$ in the k-th summand of the numerator of the fraction on the right hand side of (3.7), we get

$$\begin{aligned} &\frac{\sum_{k=1}^n S_k (c_{k+1} - c_k) 1_{\mathcal{C}}(k)}{T_n} = \\ &= \frac{\sum_{k=1}^n (c_{k+1} S_k - c_k S_{k-1}) 1_{\mathcal{C}}(k)}{T_n} - \frac{\sum_{k=1}^n b_k 1_{\mathcal{C}}(k)}{T_n}; \end{aligned} \tag{4.1}$$

Denote by

$$\tau_1 := \limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(c_{k+1}S_k - c_k S_{k-1})1_{\mathcal{C}}(k)}{T_n}. \tag{4.2}$$

From the relation

$$1_{\mathcal{C}}(k)\, c_{k+1}S_k \geq 1_{\mathcal{C}}(k)\, c_k S_k \geq 1_{\mathcal{C}}(k)\, c_k S_{k-1},$$

we deduce that $\tau_1 \geq 0$.

From relation (4.1) we get the following result, which states a condition on the finiteness of σ_1 in terms of τ_1.

Proposition 4.3. *σ_1 is finite if and only if τ_1 is, and in this case we have*

$$\underline{d}_b(\mathcal{C}) \leq \tau_1 - \sigma_1 \leq \overline{d}_b(\mathcal{C}).$$

As a consequence, if $d_b(\mathcal{C})$ exists, then

$$\tau_1 - \sigma_1 = d_b(\mathcal{C}).$$

Put

$$\tau_2 := \limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(\tilde{c}_{k+1}T_k - \tilde{c}_k T_{k-1})1_{\mathcal{D}}(k)}{S_n}.$$

Then we have the dual statement

Proposition 4.4. *σ_2 is finite if and only if τ_2 is, and in this case we have*

$$\underline{d}_a(\mathcal{D}) \leq \tau_2 - \sigma_2 \leq \overline{d}_a(\mathcal{D}).$$

As a consequence, if $d_a(\mathcal{D})$ exists, then

$$\tau_2 - \sigma_2 = d_a(\mathcal{D}).$$

We can apply Propositions 4.3 and 4.4 also to the problem of finding conditions under which $\sigma_1 = 0$ (or $\sigma_2 = 0$).

A first easy remark is that $\sigma_1 = 0$ (resp. $\sigma_2 = 0$) if $c_{n+1} = c_n$ for sufficiently large $n \in \mathcal{C}$ (resp. $n \in \mathcal{D}$) (see the definitions of σ_1 and σ_2 in (3.7) and (3.8)).

By Propositions 4.3 (resp. 4.4), a less trivial condition assuring that $\sigma_1 = 0$ (resp. $\sigma_2 = 0$) is that $d_b(\mathcal{C})$ (resp. $d_a(\mathcal{D})$) exists and $\tau_1 = d_b(\mathcal{C})$ (resp. $\tau_2 = d_a(\mathcal{D})$). Apart from the obvious case of a finite set $\mathcal{C}$ (resp. $\mathcal{D}$), it is not difficult to see that this happens for instance if $\mathcal{C}$ (resp. $\mathcal{D}$) is infinite with density $d_b(\mathcal{C})$ (resp. $d_a(\mathcal{D})$) and

$$c_{n+1}S_n - c_n S_{n-1} \sim b_n, \qquad n \to \infty, \qquad n \in \mathcal{C}, \tag{4.5}$$

(resp. $\tilde{c}_{n+1}T_n - \tilde{c}_n T_{n-1} \sim a_n$ as $n \to \infty$, $n \in \mathcal{D}$).

In order to be sure that the above set of conditions is consistent, we provide a non trivial example.

Example 4.6. Let $a_n = 1$ for every n (asymptotic density). We look for a sequence (b_n) such that (4.5) holds; by the equality $S_n = n$, in this case (4.5) becomes

$$b_{n+1}n - b_n(n-1) \sim b_n, \qquad n \to \infty, \qquad n \in \mathcal{C},$$

or, equivalently

$$\frac{b_{n+1}}{b_n} = 1 + o(1/n), \qquad n \to \infty, \qquad n \in \mathcal{C}. \tag{4.7}$$

Define

$$c_n = b_n = \begin{cases} \log n & \text{if } n \text{ is even;} \\ \log(n+1) + (\sqrt{\log n})/n & \text{if } n \text{ is odd.} \end{cases}$$

Then it is easy to see that $\mathcal{C}$ is the set of even numbers. Relation 4.7 holds, since, if n is even, we have

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{\log(n+2)}{\log n} + \frac{\sqrt{\log(n+1)}}{(n+1)\log n} = 1 + \frac{\log\big(1 + (2/n)\big)}{\log n} + \frac{\sqrt{\log(n+1)}}{(n+1)\log n} \\ &= 1 + o(1/n). \end{aligned}$$

There remains to show that $d_b(\mathcal{C})$ exists. In fact, we can even calculate $d_b(\mathcal{C})$ (proving that it is equal to 1/2). Put $r_n = [n/2]$; then we have

$$\begin{aligned} \Bigg(\sum_{\substack{1\le k\le n \\ k\in\mathcal{C}}} b_k\Bigg)\Bigg(\sum_{1\le k\le n} b_k\Bigg)^{-1} &= \frac{\sum_{k=1}^{r_n} b_{2k}}{\sum_{k=1}^{r_n} b_{2k} + \sum_{k=1}^{r_n-1} b_{2k+1}} \\ &= \Bigg(\sum_{k=1}^{r_n}\log(2k)\Bigg)\Bigg(2\sum_{k=1}^{r_n}\log(2k) - \log 2 + \sum_{k=1}^{r_n-1}\frac{\sqrt{\log(2k+1)}}{2k+1}\Bigg)^{-1}. \end{aligned}$$

Dividing both factors above by $\sum_{k=1}^{r_n}\log(2k)$, the claim follows since $\sum_k \log(2k) = +\infty$ and

$$\lim_{n\to\infty}\Bigg(\sum_{k=1}^{n-1}\frac{\sqrt{\log(2k+1)}}{2k+1}\Bigg)\Bigg(\sum_{k=1}^{n}\log(2k)\Bigg)^{-1} = \lim_{n\to\infty}\frac{\sqrt{\log(2n-1)}}{(2n-1)(\log(2n))} = 0,$$

by Cesaro's Theorem.

Going back to our original questions (in which cases is $\sigma_i < +\infty$?, in which cases is $\sigma_i = 0$?) the following Proposition 4.8 indicates another interesting approach:

Proposition 4.8. *Put*

$$\Phi_n = \frac{S_n}{a_n}, \qquad \Psi_n = \frac{T_n}{b_n}$$

and assume that

$$L_1 := \limsup_{n\to\infty} \left(\frac{c_{n+1}}{c_n} - 1\right) \Phi_n 1_{\mathcal{C}}(n) < +\infty;$$

$$\left(\text{ resp. } L_2 := \limsup_{n\to\infty} \left(\frac{c_n}{c_{n+1}} - 1\right) \Psi_n 1_{\mathcal{D}}(n) < +\infty\right).$$

Then $\sigma_1 \le L_1 \overline{d}_b(\mathcal{C})$ (resp. $\sigma_2 \le L_2 \overline{d}_a(\mathcal{D})$). In particular, $\sigma_1 = 0$ if $d_b(\mathcal{C}) = 0$ (resp. $\sigma_2 = 0$ if $d_a(\mathcal{D}) = 0$).

Proof. We consider only the case of L_1. Fix $\epsilon > 0$, and let $n_0 \in \mathbb{N}^*$ be such that, for $n > n_0$,

$$\left(\frac{c_{n+1}}{c_n} - 1\right) \Phi_n\, 1_{\mathcal{C}}(n) < L_1 + \epsilon.$$

Putting, moreover, $R_n := (1/T_n) \sum_{k=1}^{n_0} (c_{k+1} - c_k) S_k 1_{\mathcal{C}}(k)$, we observe that $\lim_{n\to\infty} R_n = 0$. Then we have, for $n > n_0$

$$\begin{aligned}
\frac{1}{T_n} \sum_{k=1}^{n} (c_{k+1} - c_k) S_k 1_{\mathcal{C}}(k) &= R_n + \frac{1}{T_n} \sum_{k=n_0+1}^{n} (c_{k+1} - c_k) S_k 1_{\mathcal{C}}(k) \\
&= R_n + \frac{1}{T_n} \sum_{k=n_0+1}^{n} \left(\frac{c_{k+1}}{c_k} - 1\right) \Phi_k b_k 1_{\mathcal{C}}(k) \\
&\le R_n + (L_1 + \epsilon) \frac{1}{T_n} \sum_{k=n_0+1}^{n} b_k 1_{\mathcal{C}}(k) \le R_n + (L_1 + \epsilon) \frac{1}{T_n} \sum_{k=1}^{n} b_k 1_{\mathcal{C}}(k),
\end{aligned}$$

and we conclude using the arbitrariness of ϵ. □

We point out some interesting applications of Proposition 4.8; the first one is concerned with the asymptotic density:

Corollary 4.9. *Assume that $\mathcal{C}$ is infinite and let (b_n) be a sequence of strictly positive numbers such that*

$$\frac{b_{n+1}}{b_n} = 1 + O(1/n), \qquad n \to \infty, \qquad n \in \mathcal{C}.$$

Then the d_b-density extends the asymptotic density.

Remark 4.10. Examples of sequences (b_n) satisfying the hypotheses of Corollary 4.9 can be constructed as follows. Define

$$b_n = \begin{cases} l_n & \text{if } n \text{ is even;} \\ l_{n+1} + (l_n/n) & \text{if } n \text{ is odd,} \end{cases}$$

where (l_n) is a fixed positive increasing sequence such that the two sequences (l_{n+1}/l_n) and $\big(n(l_{n+2} - l_n)/l_n\big)$ are bounded ($l_n = n^2$ for instance). We omit the details for the sake of brevity.

Before stating the second application of Proposition 4.8, we recall the concept of *slowly varying* function and give some of their properties.

Definition 4.11. A strictly positive function H (not necessarily monotone), defined on some half line $(a, +\infty)$ is said to be *slowly varying* as $x \to \infty$ if, for every $t > 0$, it satisfies the condition

$$\lim_{x\to\infty} \frac{H(tx)}{H(x)} = 1.$$

(see [Feller 2] for a reference, p. 276).

The following Lemmas 4.12 and 4.14 are concerned with some properties of slowly varying functions. For the proof of the first lemma, see [Feller 2], p. 282.

Lemma 4.12 (the "representation Lemma")**.** *A function H varies slowly as $x \to \infty$ iff it can be put into the form*

$$H(x) = \psi(x) \exp\left(\int_1^x \frac{\phi(t)}{t}\, dt\right),$$

where

$$\begin{cases} \lim_{x\to\infty} \psi(x) = c & \textit{with } 0 < c < \infty; \\ \lim_{x\to\infty} \phi(x) = 0. \end{cases} \tag{4.13}$$

The next lemma relates the behaviour of $H(x)$ to the behaviour of the sum $\sum_{k=1}^{n} k^\alpha H(k)$; a more complete version of it is proved in [Giuliano Antonini et al. 6] (Lemma (4.8)).

Lemma 4.14. *Let H be a slowly varying function. Then for every α, with $\alpha > -1$ we have*

$$\lim_{n\to+\infty} \frac{n^{\alpha+1} H(n)}{\sum_{k=1}^{n} k^\alpha H(k)} = \alpha + 1.$$

We are now ready to study the following situation. Let $\alpha > -1$ and $\beta > -1$ be two real numbers. Let H_1 and H_2 be two slowly varying functions; denote by ψ_1 (resp. ψ_2) the corresponding function of the representation Lemma 4.12 relative to H_1 (resp. H_2), and assume that

$$\limsup_{n\to\infty} n\,|\psi_1(n+1) - \psi_1(n)| < +\infty, \quad \text{and} \tag{4.15}$$

$$\limsup_{n\to\infty} n\,|\psi_2(n+1) - \psi_2(n)| < +\infty. \tag{4.16}$$

Remark 4.17. Since both ψ_1 and ψ_2 have strictly positive finite limits as $x \to \infty$ (see (4.13)), it is easy to see that the above assumptions imply (in fact, are equivalent to)

$$\limsup_{n\to\infty} n\left|\log\frac{\psi_1(n+1)}{\psi_1(n)}\right| < +\infty, \quad \text{and}$$

$$\limsup_{n\to\infty} n\left|\log\frac{\psi_2(n+1)}{\psi_2(n)}\right| < +\infty.$$

Moreover, by Olivier's Theorem (which states that, if $(\rho_n)_n$ is a non-increasing strictly positive sequence such that $\sum_n \rho_n < \infty$, then $\lim_{n\to\infty} n\rho_n = 0$; see [Olivier 11] for a reference) it is easy to see that, if the sequence $(\psi_i(n))_n$ is increasing and the sequence $(\psi_i(n+1) - \psi_i(n))_n$ is non–increasing, then

$$\lim_{n\to\infty} n(\psi_i(n+1) - \psi_i(n)) = \lim_{n\to\infty} n\log\frac{\psi_i(n+1)}{\psi_i(n)} = 0, \quad i = 1, 2.$$

In the setting described above, we have the following

Corollary 4.18. *Let $\alpha > -1$ and $\beta > -1$ be two real numbers and H_1 and H_2 be two slowly varying functions such that the corresponding functions ψ_1 and ψ_2 of the representation Lemma 4.12 verify (4.15) and (4.16) respectively. Consider the two sequences defined by*

$$a_n = H_1(n)n^\alpha, \qquad b_n = H_2(n)n^\beta.$$

Then the d_a-density and the d_b-density are equivalent.

Remark 4.19. The particular case of $H_1 = H_2 = 1$ is proved in [Fuchs et al. 3] by a different technique.

P r o o f. Put

$$R(n) = \frac{H_2(n)}{H_1(n)}.$$

Then $c_n = n^{\beta-\alpha}R(n)$. Moreover, with the notations of Proposition 4.8 we have, by Lemma 4.14

$$\Phi_n = \frac{\sum_{k=1}^n k^\alpha H_1(k)}{n^\alpha H_1(n)} \sim \frac{n}{\alpha+1}, \qquad n \to \infty. \tag{4.20}$$

Following Proposition 4.8, we prove that L_1 and L_2 are finite. By (4.20) we have

$$\Big(\frac{c_{n+1}}{c_n} - 1\Big)\Phi_n 1_C(n) \leq \Big|\frac{c_{n+1}}{c_n} - 1\Big|\Phi_n \sim \Big|\frac{c_{n+1}}{c_n} - 1\Big|\frac{n}{\alpha+1},$$

as $n \to \infty$. Now

$$\begin{aligned} n\Big|\frac{c_{n+1}}{c_n} - 1\Big| &= n\left|\Big(\frac{n+1}{n}\Big)^{\beta-\alpha}\frac{R(n+1)}{R(n)} - 1\right| \\ &\sim \left|n|\beta-\alpha|\log\Big(1+\frac{1}{n}\Big) + n\log\frac{R(n+1)}{R(n)}\right|. \end{aligned}$$

The first term above is clearly bounded. As to the second term, by the representation Lemma 4.12, we can write

$$\begin{aligned} &n\left|\log\frac{R(n+1)}{R(n)}\right| \\ &= \left|n\log\frac{\psi_2(n+1)}{\psi_2(n)} - n\log\frac{\psi_1(n+1)}{\psi_1(n)} + n\int_n^{n+1}\frac{\phi_2(t)-\phi_1(t)}{t}\mathrm{d}t\right| \\ &\leq n\left|\log\frac{\psi_2(n+1)}{\psi_2(n)}\right| + n\left|\log\frac{\psi_1(n+1)}{\psi_1(n)}\right| + n\left|\int_n^{n+1}\frac{\phi_2(t)-\phi_1(t)}{t}\mathrm{d}t\right|; \end{aligned}$$

now, the two first terms in the sum above are bounded according to Remark 4.17. As to the third term, by the properties of ϕ_1 and ϕ_2 (see (4.13)) we have, for sufficiently large n,

$$n\left|\int_n^{n+1}\frac{\phi_2(t)-\phi_1(t)}{t}\mathrm{d}t\right| \leq n\int_n^{n+1}\frac{1}{t}\mathrm{d}t < +\infty.$$

This shows that L_1 is finite. The argument for L_2 is identical: it is enough to interchange the roles of (a_n) and (b_n) everywhere above. □

5. Applications to the problem of the weighted density of a transformed set

Here we discuss the second application of the general comparison Theorem 2.7.

We focus our attention on the following situation. Let (a_n) be a sequence of positive numbers and $\pi : \mathbb{N}^* \to \mathbb{N}^*$ be an injective function; (a_n) and π will be fixed throughout. The aim of the present section is to establish a comparison between the d_a–densities (upper and lower) of a given set $E \subseteq \mathbb{N}^*$ and those of the transformed set $\pi(E)$ in terms of some suitable features of (a_n) and π. Put

$$e_k = \frac{a_{\pi(k)}}{a_k}, \qquad k \in \mathbb{N}^*, \quad e_0 = 0$$

and

$$\sigma'_\pi := \limsup_{n\to\infty} \frac{1}{S_n} \sum_{\pi(k+1)\le n<\pi(k)} S_k e_{k+1}; \tag{5.1}$$

$$\sigma''_\pi := \limsup_{n\to\infty} \frac{1}{S_n} \sum_{\substack{\pi(k)\le n \\ \pi(k+1)\le n}} S_k \big|e_k - e_{k+1}\big|; \tag{5.2}$$

$$\sigma_\pi := \sigma'_\pi + \sigma''_\pi. \tag{5.3}$$

Let $\pi(\mathbb{N}^*)$ be the image of π and put

$$\underline{\ell} = \underline{d}_a(\pi(\mathbb{N}^*)); \qquad \overline{\ell} = \overline{d}_a(\pi(\mathbb{N}^*)).$$

We shall prove the following

Theorem 5.4. *(i) Assume that $\sigma_\pi < +\infty$. Then*

$$\sigma_\pi(\underline{d}_a(E) - \overline{d}_a(E)) + \underline{\ell}\,\underline{d}_a(E) \le \underline{d}_a(\pi(E)) \le \overline{d}_a(\pi(E)) \le \sigma_\pi(\overline{d}_a(E) - \underline{d}_a(E)) + \overline{\ell}\,\overline{d}_a(E).$$

(ii) Assume that the sequence (e_n) is non–increasing, and that

$$\sigma'''_\pi := \limsup_{n\to\infty} \frac{1}{S_n} \sum_{\pi(k+1)\le n<\pi(k)} S_k < +\infty.$$

Then $\sigma'_\pi < +\infty$ and

$$\sigma'_\pi(\underline{d}_a(E) - \overline{d}_a(E)) + \underline{\ell}\,\underline{d}_a(E) \le \underline{d}_a(\pi(E)) \le \overline{d}_a(\pi(E)) \le \sigma'_\pi(\overline{d}_a(E) - \underline{d}_a(E)) + \overline{\ell}\,\overline{d}_a(E).$$

The following Corollary is an immediate consequence of Theorem 5.4.

Corollary 5.5. *In addition to the assumptions of Theorem 5.4 (i) or those of Theorem 5.4 (ii), suppose that $\pi(\mathbb{N}^*)$ has a d_a-density equal to ℓ. Then, if $d_a(E)$ exists, then also $d_a(\pi(E))$ exists and*

$$d_a(\pi(E)) = \ell\, d_a(E).$$

Remark 5.6. In the recent paper [Nathanson et al. 9] the authors prove a result in the same direction as our Theorem 5.4 but with different assumptions. It must be noted that they take into account only the particular case $a_n = 1$, $\forall\, n$ (i.e. the considered a-density is the asymptotic density, called d); moreover they assume that $\underline{l} = \overline{l} = 1$ (i.e. $d(\pi(\mathbb{N}^*))$ exists and is equal to 1) and that $\underline{d}(E) = \overline{d}(E)$ (i.e. $d(E)$ exists). Hence in this sense our result is more complete (see Corollary 5.8 below). Moreover it enlightens the fact that the densities (upper and lower) of the transformed set $\pi(E)$ are governed in some sense by the sets of "inversions" of π (see the definitions of the sets D_r, E_r, F_r, G_r in the proof of Theorem 5.4).

Remark 5.7. (i) Let $\mathcal{G} = \{\pi : \pi \text{ permutation of } \mathbb{N}^*, \sigma_\pi < +\infty\}$. Since in this case $\ell = d_a(\pi(\mathbb{N}^*)) = d_a(\mathbb{N}^*) = 1$, the above Corollary implies that every $\pi \in \mathcal{G}$ preserves the d_a density. In fact, if $\overline{d}_a(E) = \underline{d}_a(E) = d_a(E)$, Theorem 5.4 (i) gives (by substitution)

$$d_a(E) \le \underline{d}_a(\pi(E)) \le \overline{d}_a(\pi(E)) \le d_a(E),$$

hence $d_a(\pi(E))$ exists and is equal to $d_a(E)$.
(ii) If $(e_n)_n$ is non–increasing, the same is true for every permutation π belonging to $\mathcal{G}' = \{\pi : \pi \text{ permutation of } \mathbb{N}^*, \sigma'_\pi < +\infty\}$. This follows from Theorem 5.4 (ii).

We point out the following particular case of Corollary 5.5, concerning the case of the asymptotic density:

Corollary 5.8. *Let π be a permutation of the integers such that $\underline{l} = \overline{l} = 1$ and*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{\pi(k+1)\le n<\pi(k)} k < +\infty.$$

Then π preserves the asymptotic density.

P r o o f o f T h e o r e m 5.4. We rephrase our problem in the setting of infinite matrices, in order to apply the general comparison Theorem 2.7.

Concerning the function π, we make the following remarks. Any fixed integer n can be identified with the indicator function $1_{\{n\}}$, which in turn can be viewed as the infinite (column) vector $v^{(n)}$ of its values: more precisely

$$v^{(n)} = \big(v_k^{(n)}\big)_k, \qquad \text{with} \quad v_k^{(n)} = 1_{\{n\}}(k) = \delta_{k,n}. \tag{5.9}$$

Here and in the sequel $\delta_{k,n}$ is the usual Kronecker symbol.

Consider the infinite matrix $\mathbf{M}_\pi = (m_{h,k})_{h,k}$, with

$$m_{h,k} = \delta_{h,\pi(k)}. \tag{5.10}$$

We can associate $\mathbf{M}_\pi$ to π in the sense that $\pi(n) = j$ if and only if $\mathbf{M}_\pi v^{(n)} = v^{(j)}$. This is justified by the following chain of equalities, where we use relations (5.9) (twice) and (5.10):

$$\left(\mathbf{M}_\pi v^{(n)}\right)_h = \sum_k m_{h,k} v_k^{(n)} = \sum_k \delta_{h,\pi(k)} \delta_{k,n} = \delta_{h,\pi(n)} = v_h^{(\pi(n))} = v_h^{(j)},$$

($h \in \mathbb{N}^*$). The above relation simply means that

$$\mathbf{M}_\pi v^{(n)} = v^{(\pi(n))},$$

and can easily be extended by linearity to any set $E \subseteq \mathbb{N}^*$, by noting that the column vector $v^{(E)}$ corresponding to 1_E has the representation

$$v^{(E)} = 1_E = \sum_{n \in E} 1_{\{n\}} = \sum_{n \in E} v^{(n)},$$

whence

$$\mathbf{M}_\pi v^{(E)} = \sum_{n \in E} \mathbf{M}_\pi v^{(n)} = \sum_{n \in E} v^{(\pi(n))} = \sum_{j \in \pi(E)} v^{(j)} = v^{(\pi(E))},$$

since π is injective.

In order to have less cumbersome notations, we shall write the above (with a slight abuse of notation) as

$$\mathbf{M}_\pi 1_E = 1_{\pi(E)}. \tag{5.11}$$

Let now $\mathbf{A} = (a_{n,k})$ be the Riesz matrix associated to the sequence (a_n), according to Definition 2.15. The matrix $\mathbf{A}$ defines the d_a–weighted density, as explained in Section 3. For future use, we remark that $a_{n,k}$ can be written in the more compact form

$$a_{n,k} = \frac{a_k}{S_n} 1_{\{1,2,\ldots,n\}}(k), \tag{5.12}$$

where, as in Section 2, we put $S_n = \sum_{k=1}^n a_k$.

Consider the inverse matrix $\mathbf{A}^{-1} = (\tilde{a}_{h,k})$ (see Section 2). We rewrite here the formula for $\tilde{a}_{h,k}$ given in Section 2 (see (2.16)) as

$$\tilde{a}_{h,k} = (-1)^{h-k} \frac{S_k}{a_h} 1_{\{k,k+1\}}(h). \tag{5.13}$$

Relation (5.11) implies that

$$\underline{d}_a(\pi(E)) = \liminf_{n \to \infty} \left(\mathbf{A} 1_{\pi(E)}\right)_n = \liminf_{n \to \infty} \left((\mathbf{A}\mathbf{M}_\pi) 1_E\right)_n,$$

and an analogous relation holds for $\overline{d}_a(\pi(E))$. Hence we see that we can apply our general Theorem 2.7 with $\mathbf{U} = \mathbf{A}$ and $\mathbf{V} = \mathbf{AM}_\pi$. This means that we must study the matrix $\mathbf{W} = \mathbf{VU}^{-1} = \mathbf{AM}_\pi\mathbf{A}^{-1}$ and calculate $\sigma_{\mathbf{W}}$.

So, put $\mathbf{AM}_\pi\mathbf{A}^{-1} = (d_{r,n})$; we shall first calculate $d_{r,n}$ in terms of π and (a_n). We shall use the representations (5.10), (5.12) and (5.13). We have

$$\begin{aligned} d_{r,n} &= \sum_{h,k} a_{r,h} m_{h,k} \tilde{a}_{k,n} = \sum_k \tilde{a}_{k,n} \sum_h a_{r,h} m_{h,k} \\ &= \sum_k (-1)^{k-n} \frac{S_n}{a_k} 1_{\{n,n+1\}}(k) \sum_h \frac{a_h}{S_r} 1_{\{1,2,\ldots,r\}}(h) \delta_{h,\pi(k)} \\ &= \sum_k (-1)^{k-n} \frac{S_n}{a_k} 1_{\{n,n+1\}}(k) \frac{a_{\pi(k)}}{S_r} 1_{\{1,2,\ldots,r\}}(\pi(k)) \\ &= \frac{S_n}{S_r} \sum_k (-1)^{k-n} e_k 1_{\{n,n+1\}}(k) 1_{\{1,2,\ldots,r\}}(\pi(k)). \end{aligned} \tag{5.14}$$

The (α, β) condition in Theorem 2.7 is concerned with the lim inf and the lim sup of $\sum_n d_{r,n}$ as $r \to \infty$. We get from the above calculation

$$\begin{aligned} \sum_n d_{r,n} &= \frac{1}{S_r} \sum_n S_n \sum_k (-1)^{k-n} e_k 1_{\{n,n+1\}}(k) 1_{\{1,2,\ldots,r\}}(\pi(k)) \\ &= \frac{1}{S_r} \sum_k e_k 1_{\{1,2,\ldots,r\}}(\pi(k)) \sum_n (-1)^{k-n} S_n 1_{\{n,n+1\}}(k) \\ &= \frac{1}{S_r} \sum_k e_k 1_{\{1,2,\ldots,r\}}(\pi(k)) \sum_{n=k-1,k} (-1)^{k-n} S_n \\ &= \frac{1}{S_r} \sum_k e_k 1_{\{1,2,\ldots,r\}}(\pi(k)) \big(S_k - S_{k-1}\big) \\ &= \frac{1}{S_r} \sum_k e_k 1_{\{1,2,\ldots,r\}}(\pi(k)) a_k \\ &= \frac{1}{S_r} \sum_k a_{\pi(k)} 1_{\{1,2,\ldots,r\}}(\pi(k)) = \frac{\sum_{k=1}^r a_k 1_{\pi(\mathbb{N}^*)}(k)}{S_r}. \end{aligned} \tag{5.15}$$

Hence we have

$$\alpha = \liminf_{r\to\infty} \sum_n d_{r,n} = \underline{\ell}; \qquad \beta = \limsup_{r\to\infty} \sum_n d_{r,n} = \overline{\ell}. \tag{5.16}$$

In order to prove that $\mathbf{W} = \mathbf{A}\mathbf{M}_\pi\mathbf{A}^{-1}$ is regular and calculate $\sigma_{\mathbf{W}}$, we write $d_{r,n}$ more explicitly. For simplicity we put, for every integer r,

$$\begin{aligned} D_r &= \{k \in \mathbb{N}^* : \pi(k) \le r, \pi(k+1) \le r\}; \\ E_r &= \{k \in \mathbb{N}^* : \pi(k+1) \le r < \pi(k)\}; \\ F_r &= \{k \in \mathbb{N}^* : \pi(k) \le r < \pi(k+1)\}; \\ G_r &= \{k \in \mathbb{N}^* : \pi(k) > r, \pi(k+1) > r\}. \end{aligned}$$

Then it is easy to see that the last expression of relation (5.14) is equal to

$$\frac{S_n}{S_r}\sum_k (-1)^{k-n} e_k 1_{\{n,n+1\}}(k) 1_{\pi^{-1}(\{1,2,\dots,r\})}(k)$$

and this means that

$$d_{r,n} = \begin{cases} \frac{S_n}{S_r}\big\{e_n - e_{n+1}\big\} & \text{if } n \in D_r \\ \frac{S_n}{S_r}\, e_n & \text{if } n \in F_r \\ -\frac{S_n}{S_r}\, e_{n+1} & \text{if } n \in E_r \\ 0 & \text{if } n \in G_r. \end{cases} \tag{5.17}$$

The regularity of $\mathbf{W}$ is now immediate since $S_r \to \infty$ as $r \to \infty$ (though not relevant, we remark that, n being fixed, $\pi(n) \le r$ and $\pi(n+1) \le r$ for sufficently large r, hence $n \in D_r$).

As to $\sigma_{\mathbf{W}}$, we observe that $d_{r,n} \le 0$ if $n \in E_r$ and $d_{r,n} \ge 0$ if $n \in F_r$ or $n \in G_r$, while, if $n \in D_r$, $d_{r,n}$ has the same sign as $e_n - e_{n+1}$; these remarks lead to the equality $\sigma_{\mathbf{W}} = \sigma_\pi$ and prove Theorem 5.4 (i).

In order to prove Theorem 5.4 (ii), it remains only to observe that, in this case, we need not consider σ''_π since $e_n - e_{n+1} \ge 0$ for every n; moreover, e_n being bounded by e_1, $\sigma'_\pi < +\infty$ if $\sigma'''_\pi < +\infty$. ☐

Remark 5.18. As we have just seen in formulas (5.17), $\mathbf{W}$ may indeed have some negative terms; hence it is important not to assume positivity in our general comparison Theorem 2.7.

Remark 5.19. Referring to Theorem 5.4 (ii), we see that, by its very definition, E_r (defined during the proof of the Theorem) is formed of *isolated* integers (i.e., if $k \in E_r$, then neither $k+1$ nor $k-1$ belong to E_r). Moreover E_r is a finite set (otherwise π could not be injective). Hence we can write

$$E_r = \{u_{1,r}, u_{2,r}, \dots, u_{t_r,r}\}$$

for suitable integers t_r and $u_{j,r}$, and

$$\sum_{\pi(k+1)\le r<\pi(k)} S_k = \sum_{k\in E_r} S_k = \sum_{i=1}^{t_r} S_{u_{i,r}} \le t_r S_{u_{t_r,r}}.$$

Hence, if $(t_r)_r$ and $(S_{u_{t_r},r}/S_r)$ are bounded sequences, we have $\sigma_\pi''' < \infty$. The next example shows such a situation.

Example 5.20. We consider the permutation π that interchanges any odd number with its successor (1 with 2, 3 with 4 and so on), i.e. in formula, $\pi(k) = k + (-1)^{k-1}$.

Let (a_n) be such that, for every odd integer n, we have $a_n = a_{n+1}$. Then it is immediate to see that $e_n = 1$ for each n, hence the first condition of (5.4) (ii) is satisfied. Moreover, for every r, we have $E_r = \{r\}$, hence (using the notations of Remark 5.19), we easily get $t_r = 1$ and $u_{t_r,r} = r$ for every r.

Remark 5.21. Example 5.20 shows that the set of pairs $(\pi, (a_n))$ verifying the assumptions of Theorem 5.4 is not empty. Moreover it is interesting since it contains the case of asymptotic density as a particular one, and we get the following result: the permutation considered in Example 5.20 does not change the asymptotic density of any set E.

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